SOI-KF: DISTRIBUTED KALMAN FILTERING WITH LOW-COST COMMUNICATIONS USING THE SIGN OF INNOVATIONS*

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ABSTRACT

We derive and analyze distributed state estimators of dynamical stochastic processes, whereby low communication cost is effected by requiring the transmission of a single bit per observation. Following a Kalman filtering (KF) approach, we develop recursive algorithms for distributed state estimation based on the sign of innovations (SOI). Even though SOI-KF can afford minimal communication overhead, we prove that in terms of performance and complexity it comes very close to the clairvoyant KF which is based on the analog-amplitude observations. Reinforcing our conclusions, we show that the SOI-KF applied to distributed target tracking based on distance only observations yields accurate estimates at low communication cost.

1. INTRODUCTION

Distributed signal processing is a well-appreciated toolbox for decentralized tracking applications involving e.g., multiple radars, but has received a revived interest in the context of wireless sensor networks (WSNs). Unlike centralized signal processing, observations and the resultant algorithms are physically distributed across sensors in the network, dictating that inter-sensor communications should be viewed as an integral part of the problem at hand. For the problem of distributed estimation of dynamical stochastic processes considered in this paper, communications dictate estimation to be based on quantized observations – a problem certainly different from state estimation based on the original (analog-amplitude) observations.

Quantizer design for WSNs was studied in [5], where the concept of information loss was defined as the relative increase in estimation variance when analog-amplitude observations are replaced by their quantized versions. To address the challenge of building suitable noise models for WSNs, universal estimators that work irrespective of the noise distribution were introduced in [3, 6]. Estimating signals using very noisy sensor data was studied in [6], where it was shown that as the noise variance becomes comparable with the parameter's dynamic range, quantization to a single bit per observation leads to low complexity estimators of *time-invariant deterministic* parameters with minimal information loss.

Accounting for the stringent bandwidth constraints of WSNs, we study state estimation of *dynamical stochastic* processes based on severely quantized observations, whereby low-cost communications restrict sensors to transmit a single bit per observation. The quantization rule manifests itself in a non-linear measurement equation in a Kalman Filtering (KF) setup. While the discontinuous non-linearity precludes application of the extended (E)KF, one could use more powerful techniques such as the unscented (U)KF [1], or the Particle Filter (PF) [2]. However, since there are no tools to predict the performance of the UKF and the PF, no insight has been provided with regards to performance degradation when quantized data are used in lieu of the analog-amplitude observations; furthermore, these approaches are significantly more complex than (E)KF. Our contribution is to *design state estimators based on binary observations* so that: i) complexity is rendered comparable to the equivalent KF based on the original observations; and, ii) the mean squared error (MSE) of the resultant estimator based on binary observations is close to the MSE of the estimator based on the original observations.

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Notation: We use $p(x|y)$ to denote the probability density function (pdf) of the random variable (r.v.) x given the r.v. y. A normally distributed r.v. with mean $\boldsymbol{\mu}_x = \mathrm{E}(\mathbf{x})$ and covariance matrix $\mathbf{C}_x = \mathrm{E}(\mathbf{x}\mathbf{x}^T)$, is written as $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_x, \mathbf{C}_x)$. For a scalar r.v., we write $p(x) = \mathcal{N}(x; \mu_x, \sigma_x^2)$ and define $Q(x) := \int_x^\infty \mathcal{N}(u; 0, 1)du$. We use $\delta_c(t)$ to denote the Dirac delta function and $\delta(n)$ to denote the Kronecker delta. Lower (upper) case boldface letters will stand for column vectors (matrices) and $\mathbf{I}$ will denote the identity matrix .

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider an ad-hoc WSN (see Fig. 1) with K distributed sensors $\{S_k\}_{k=1}^K$ deployed to track a $p \times 1$ real random vector (state) $\mathbf{x}_c(t) \in \mathbb{R}^p$. The state evolution in continuous-time is described by

$$\dot{\mathbf{x}}_c(t) = \mathbf{A}_c(t)\mathbf{x}_c(t) + \mathbf{u}_c(t), \tag{1}$$

where $\mathbf{A}_c(t) \in \mathbb{R}^{p\times p}$, and the driving input $\mathbf{u}_c(t) \in \mathbb{R}^p$ is a zero-mean white Gaussian process with autocorrelation $\mathrm{E}[\mathbf{u}_c(t_1)\mathbf{u}_c^T(t_2)] = \mathbf{C}_{u_c}(t_1)\delta_c(t_1 - t_2)$. We model the observation $\mathbf{y}_c(t, k) \in \mathbb{R}^M$ of Sensor S_k at time t as

$$\mathbf{y}_c(t, k) = \mathbf{H}_c(t, k)\mathbf{x}_c(t) + \mathbf{v}_c(t, k), \tag{2}$$

where $\mathbf{H}_c(t, k) \in \mathbb{R}^{M\times p}$ and the observation noise $\mathbf{v}_c(t, k) \in \mathbb{R}^M$ is a zero-mean Gaussian process uncorrelated across time and sensors; i.e., $\mathrm{E}[\mathbf{v}_c(t_1, k_1)\mathbf{v}_c^T(t_2, k_2)] = \mathbf{C}_{v_c}(t_1, k_1)\delta_c(t_1-t_2)\delta(k_1-k_2)$.

To track $\mathbf{x}_c(t)$, we consider uniform sampling with period T_s and define $\mathbf{x}(n) := \mathbf{x}_c(nT_s)$ and $\mathbf{y}(n, k) := \mathbf{y}_c(nT_s, k)$ as the discrete-time state and observation vectors, respectively. From (1) and (2) we can obtain a discrete-time model [4, Sec. 4.9]. Upon defining $\boldsymbol{\Phi}(t_2, t_1) := \exp[\int_{t_1}^{t_2} \mathbf{A}_c(t)dt]$, we solve the differential equation in (1) between $(n-1)T_s$ and nT_s with initial condition $\mathbf{x}(n-1)$ to obtain the vector time-varying auto-regressive (AR) process

$$\begin{aligned}\mathbf{x}(n) &= \mathbf{A}(n)\mathbf{x}(n-1) + \mathbf{u}(n)\\ \mathbf{y}(n, k) &= \mathbf{H}(n, k)\mathbf{x}(n) + \mathbf{v}(n, k),\end{aligned} \tag{3}$$

where the matrices are $\mathbf{A}(n) := \boldsymbol{\Phi}(nT_s, (n-1)T_s)$, $\mathbf{H}(n, k) := \mathbf{H}_c(nT_s, k)$, the driving input is $\mathbf{u}(n) := \int_{(n-1)T_s}^{nT_s} \boldsymbol{\Phi}(nT_s, \tau)\mathbf{u}_c(\tau)d\tau$ and the observation noise is white Gaussian with pdf $p[\mathbf{v}(n, k)] = \mathcal{N}[\mathbf{v}(n, k); \mathbf{0}, \mathbf{C}_v(n, k)]$. Since sampling (2) requires passing $\mathbf{y}_c(t, k)$ through a low- or band-pass filter of bandwidth $1/T_s$, the sampled co-

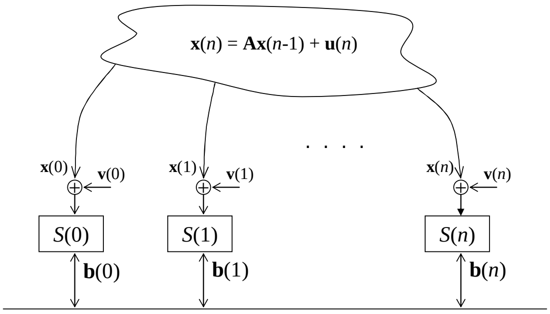

Fig. 1. Ad hoc WSN for tracking the state $\mathbf{x}(n)$

variance matrix satisfies $\mathbf{C}_v(n,k) = \mathbf{C}_{v_c}(nT_s,k)/T_s$ [4, Sec. 4.9].

Supposing that $\mathbf{A}(n)$, $\mathbf{C}_u(n)$, $\mathbf{H}(n,k)$ and $\mathbf{C}_v(n,k)$ are available to all sensors for all n,k, the goal of the WSN is for each sensor S_k to form an estimate of $\mathbf{x}(n)$. Without loss of generality, we assume that sensors convey their observations to each other using time division multiple access (TDMA) with $k(n)$ being the index of the sensor scheduled at the n^{th} time slot; for notational brevity we denote $S_{k(n)} = S(n)$ and $\mathbf{y}(n,k(n)) = \mathbf{y}(n)$. Digital transmission requires a quantization function $\mathbf{q}_n$, to map $\mathbf{y}(n)$ into binary data:

$$\mathbf{b}(n) := \mathbf{q}_n(\mathbf{y}(n)), \quad \text{with} \quad \mathbf{q}_n : \mathbb{R}^M \to \{-1,1\}^M, \qquad (4)$$

where $\mathbf{b}(n) := [b(n,1),\ldots,b(n,M)]^T$ is an $M \times 1$ binary message. We further suppose that $\mathbf{b}(n)$ is correctly received by all sensors.

Given messages $\mathbf{b}_{0:n} := [\mathbf{b}^T(0),\ldots,\mathbf{b}^T(n)]^T$, our goal is to derive and analyze the performance of the MMSE estimator

$$\hat{\mathbf{x}}(n|n) := \mathrm{E}[\mathbf{x}(n)|\mathbf{b}_{0:n}] = \int_{\mathbb{R}^p} \mathbf{x}(n) p[\mathbf{x}(n)|\mathbf{b}_{0:n}] d\mathbf{x}(n), \qquad (5)$$

which is to be computed by all the sensors in the network. We also define the so called predictors that estimate (predict) the state and observation vectors based on past observations:

$$\begin{aligned} \hat{\mathbf{x}}(n|n-1) &:= \mathrm{E}[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathbf{A}(n)\hat{\mathbf{x}}(n-1|n-1) \quad (6) \\ \hat{\mathbf{y}}(n|n-1) &:= \mathrm{E}[\mathbf{y}(n)|\mathbf{b}_{0:n-1}] = \mathbf{H}(n)\hat{\mathbf{x}}(n|n-1). \end{aligned}$$

For the state estimators in (5) and (6), we define the error covariance matrices (ECM) $\mathbf{M}(n|n) := \mathrm{E}[(\hat{\mathbf{x}}(n|n) - \mathbf{x}(n))(\hat{\mathbf{x}}(n|n) - \mathbf{x}(n))^T]$ and $\mathbf{M}(n|n-1) := \mathrm{E}[(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n))(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n))^T]$. The two are related by the recursion

$$\mathbf{M}(n|n-1) = \mathbf{A}(n)\mathbf{M}(n-1|n-1)\mathbf{A}^T(n) + \mathbf{C}_u(n), \qquad (7)$$

which we will use in later derivations. Note that the relations in (6) and (7) hold true regardless of the quantization rule in (4).

3. STATE ESTIMATION WITH SIGN OF INNOVATIONS

We start by considering the case of scalar $\mathbf{y}(n) \leftrightarrow y(n)$ observations in (3) and define the message $b(n)$ as the *sign of innovation* (SOI):

$$b(n) = \mathrm{sign}[\tilde{y}(n)] := \begin{cases} +1, & \text{if} \quad y(n) \geq \hat{y}(n|n-1) \\ -1, & \text{if} \quad y(n) < \hat{y}(n|n-1) \end{cases}, \qquad (8)$$

where $\tilde{y}(n) := y(n) - \hat{y}(n|n-1)$ Notice that the SOI $b(n)$ is *not* a standard quantizer of the data $y(n)$. It can be thought as one with judiciously setting the quantization threshold at the data prediction $\hat{y}(n|n-1)$. Computing the MMSE $\hat{\mathbf{x}}(n|n)$ in (5) requires computationally demanding numerical integration [7]. However, using the approximation $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] \doteq \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1), \mathbf{M}(n|n-1)]$, which is standard in non-linear filtering (see e.g., [2]), we obtain a KF-like recursion described in the following proposition[1],

Proposition 1 *Consider the model in* (3) *with* $\mathbf{y}(n) \leftrightarrow y(n)$ *scalar and denote* $\mathbf{C}_v(n) \leftrightarrow \sigma_v^2(n)$ *and* $\mathbf{H}(n) \leftrightarrow \mathbf{h}^T(n) \in \mathbb{R}^{1\times p}$*. If we define binary observations as in* (8) *and assume that* $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1), \mathbf{M}(n|n-1)]$*, then the MMSE estimator* $\hat{\mathbf{x}}(n|n)$ *can be obtained as*

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{(\sqrt{2/\pi})\mathbf{M}(n|n-1)\mathbf{h}(n)}{\sqrt{\mathbf{h}^T(n)\mathbf{M}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}}\; b(n) \quad (9)$$

$$\mathbf{M}(n|n) = \mathbf{M}(n|n-1) - \frac{(2/\pi)\mathbf{M}(n|n-1)\mathbf{h}(n)\mathbf{h}^T(n)\mathbf{M}(n|n-1)}{\mathbf{h}^T(n)\mathbf{M}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}. \quad (10)$$

The assumption $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] = \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1), \mathbf{M}(n|n-1)]$ yields the low-complexity SOI-KF that implements distributed state estimation based on single bit observations using the recursions (6)-(7) and (9)-(10). To estimate $\mathbf{x}(n)$, we only require a few basic algebraic operations per iteration. Moreover, the SOI-KF recursion is strikingly reminiscent of the KF recursions (for the latter see e.g., [4, Section 4.2]). The covariance updates in particular are identical except for the $2/\pi$ factor in (10).

To implement the SOI-KF we run two independent algorithms. The observation-transmission algorithm is run by the sensors as dictated by the scheduler. The sensor starts by collecting the observation $y(n,k) \leftrightarrow y(n)$ and computing the state and observation predicted estimates $\hat{\mathbf{x}}(n|n-1)$ and $\hat{y}(n|n-1)$. Based on the latter, it obtains the SOI by means of (8) which it percolates to the remaining sensors as the message $b(n)$. The reception-estimation algorithm is continuously ran by *all* sensors to track $\mathbf{x}(n)$ and is (surprisingly) identical to a KF algorithm except for the (minor) differences in the update equations. At each time slot, we compute $\hat{\mathbf{x}}(n|n-1)$ by means of (6) to then receive the SOI $b(n)$ that we use to compute $\hat{\mathbf{x}}(n|n)$ by means of (9). A couple of remarks are now in order.

Remark 1 It is possible to express the SOI-KF corrector in (9) in a form that exemplifies its link with the KF corrector [4]. Indeed, if we define the SOI-KF innovation as

$$\tilde{b}(n) := \sqrt{(2/\pi)\left[\mathbf{h}^T(n)\mathbf{M}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)\right]}\; b(n), \qquad (11)$$

we can re-write the SOI-KF corrector as

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \frac{\mathbf{M}(n|n-1)\mathbf{h}(n)}{\mathbf{h}^T(n)\mathbf{M}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)}\; \tilde{b}(n). \quad (12)$$

Note that (12) is identical to the KF corrector if we replace $\tilde{b}(n) \leftrightarrow \tilde{y}(n)$. Moreover, note that the units of $\tilde{b}(n)$ and $\tilde{y}(n)$ are the same, and that $\mathrm{E}[\tilde{b}(n)] = \mathrm{E}[\tilde{y}(n)] = 0$. Even more interesting,

$$\mathrm{E}[\tilde{b}^2(n)] = \frac{2}{\pi}\left[\mathbf{h}^T(n)\mathbf{M}(n|n-1)\mathbf{h}(n) + \sigma_v^2(n)\right] = \frac{2}{\pi}\mathrm{E}[\tilde{y}^2(n)]$$

which explains the $(2/\pi)$ relationship between the ECM corrections for the KF and for the SOI-KF in (10). The difference between the KF corrector and (12) is that in the SOI-KF the magnitude of the correction at each step is determined by $\mathrm{E}[\tilde{b}^2(n)]$ and it is the same regardless of how large or small the actual innovation $\tilde{y}(n)$ is.

Remark 2 We have shown that as $\sigma_v^2 \to \infty$, $p[\mathbf{x}(n)|\mathbf{b}_{0:n}]$ converges uniformly to a normal distribution [7]. Thus, the assumption $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}] \doteq \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1), \mathbf{M}(n|n-1)]$ holds asymptotically as $\sigma_v^2 \to \infty$.

[1] Proofs of claims in this paper can be found in [7].

Algorithm 1 SOI-KF – Observation-transmission

Require: $\hat{\mathbf{x}}(n-1|n-1)$ and $\mathbf{M}(n-1|n-1)$
Ensure: $\mathbf{b}(n)$
1: Compute $\hat{\mathbf{x}}(n|n-1)$ and $\mathbf{M}(n|n-1)$ using (6) and (7).
2: $\mathbf{y}_0(n) := \mathbf{C}_v^{-1/2}(n)\mathbf{y}(n)$ and $\mathbf{H}_0(n) := \mathbf{C}_v^{-1/2}(n)\mathbf{H}(n)$
3: **for** $m = 1$ to M **do**
4: Compute $\hat{y}(n, m|n-1, m-1)$ using (16)
5: Compute $b(n, m)$ using (17)
6: Compute $\mathbf{k}(n, m)$, $\hat{\mathbf{x}}(n|n-1, m)$, and $\mathbf{M}(n|n-1, m)$ using (18) , (19) and (20)
7: **end for**
8: Transmit $\mathbf{b}(n) = [b(n, 1), \ldots, b(n, M)]$

Algorithm 2 SOI-KF – Reception-estimation

Require: prior estimate $\hat{\mathbf{x}}(-1|-1)$ and ECM $\mathbf{M}(-1|-1)$
1: **for** $n = 0$ to ∞ **do** {repeat for the life of the network}
2: Compute $\hat{\mathbf{x}}(n|n-1)$ and $\mathbf{M}(n|n-1)$ using (6) and (7).
3: $\mathbf{H}_0(n) := \mathbf{C}_v^{-1/2}(n)\mathbf{H}(n)$
4: Receive $\mathbf{b}(n)$
5: **for** $m = 1$ to M **do**
6: Compute $\mathbf{k}(n, m)$, $\hat{\mathbf{x}}(n|n-1, m)$, and $\mathbf{M}(n|n-1, m)$ using (18) , (19) and (20)
7: **end for**
8: $\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1, M)$, $\mathbf{M}(n|n) = \mathbf{M}(n|n-1, M)$
9: **end for**

3.1. Vector state - vector observation SOI-KF

To address the general model in (3) we define the whitened observations $\mathbf{y}_0(n) := \mathbf{C}_v^{-1/2}(n)\mathbf{y}(n)$, and rewrite (3) as

$$\mathbf{y}_0(n) = \mathbf{C}_v^{-\frac{1}{2}}(n)\mathbf{H}(n)\mathbf{x}(n) + \mathbf{C}_v^{-\frac{1}{2}}(n)\mathbf{v}(n) := \mathbf{H}_0(n)\mathbf{x}(n) + \mathbf{v}_0(n), \tag{13}$$

Upon defining $\mathbf{y}_0(n) := [y_0(n, 1), \ldots, y_0(n, M)]^T$, $\mathbf{v}_0(n) := [v_0(n, 1), \ldots, v_0(n, M)]^T$ and $\mathbf{H}_0(n) := [\mathbf{h}_0(n, 1), \ldots, \mathbf{h}_0(n, M)]^T$, we rewrite (13) componentwise as:

$$y_0(n, m) = \mathbf{h}_0^T(n, m)\mathbf{x}(n) + v_0(n, m), \;\; m \in [1, M], \tag{14}$$

where $\sigma_{v_0}^2 := \mathrm{E}[v_0^2(n, m)] = 1$. Mimicking steps in Section 3, we define $\mathbf{b}(n, 1:m) := [b(n, 1), \ldots, b(n, m)]^T$ and introduce

$$\hat{\mathbf{x}}(n|n-1, m) = \mathrm{E}[\mathbf{x}(n)|\mathbf{b}_{0:n-1}, \mathbf{b}(n, 1:m)], \tag{15}$$

which is the MMSE estimator based on past messages and the first m components of the current message. We adopt the convention $\hat{\mathbf{x}}(n|n-1, 0) = \hat{\mathbf{x}}(n|n-1)$, and note that $\hat{\mathbf{x}}(n|n-1, M) = \hat{\mathbf{x}}(n|n)$ with $\hat{\mathbf{x}}(n|n-1)$ as in (6) and $\hat{\mathbf{x}}(n|n)$ as in (5). From (15), we obtain the MMSE predictor of $y_0(n, m)$ as [c.f. (14) and (15)]

$$\begin{aligned}\hat{y}_0(n, m|n-1, m-1) &:= \mathrm{E}[y_0(n, m)|\mathbf{b}_{0:n-1}, \mathbf{b}(n, 1:m-1)] \\ &= \mathbf{h}_0^T(n, m)\hat{\mathbf{x}}(n|n-1, m-1).\end{aligned} \tag{16}$$

From (16), we define the SOI messages as

$$b(n, m) := \mathrm{sign}[y_0(n, m) - \hat{y}_0(n, m|n-1, m-1)], \tag{17}$$

for $m \in [1, M]$, to introduce the SOI-KF for vector observations:

Proposition 2 *Consider the model in* (3)*, binary observations as in* (17) *and let* $\mathbf{H}_0(n) := [\mathbf{h}_0(n, 1), \ldots, \mathbf{h}_0(n, M)]^T$ *be defined as* $\mathbf{H}_0(n) := \mathbf{C}_v^{-1/2}(n)\mathbf{H}(n)$ *[c.f. (13)]. If* $p[\mathbf{x}(n)|\mathbf{b}_{0:n-1}, \mathbf{b}(n, 1:m-1)] = \mathcal{N}[\mathbf{x}(n); \hat{\mathbf{x}}(n|n-1, m-1), \mathbf{M}(n|n-1, m-1)]$*, then the MMSE estimate* $\hat{\mathbf{x}}(n|n)$ *can be obtained from*

$$\mathbf{k}(n, m) = \frac{(\sqrt{2/\pi})\mathbf{M}(n|n-1, m-1)\mathbf{h}_0(n, m)}{\sqrt{1 + \mathbf{h}_0^T(n, m)\mathbf{M}(n|n-1, m-1)\mathbf{h}_0(n, m)}} \tag{18}$$

$$\hat{\mathbf{x}}(n|n-1, m) = \hat{\mathbf{x}}(n|n-1, m-1) + \mathbf{k}(n, m)b(n, m) \tag{19}$$

$$\mathbf{M}(n|n-1, m) = \mathbf{M}(n|n-1, m-1) - \mathbf{k}(n, m)\mathbf{k}^T(n, m). \tag{20}$$

For each time index n*,* (18) *to* (20) *are repeated for* $m \in [1, M]$*. We adopt the conventions* $\hat{\mathbf{x}}(n|n-1, 0) \equiv \hat{\mathbf{x}}(n|n-1)$ *and* $\mathbf{M}(n|n-1, 0) \equiv \mathbf{M}(n|n-1)$*, and note that the MMSE estimate and the ECM are* $\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1, M)$ *and* $\mathbf{M}(n|n) = \mathbf{M}(n|n-1, M)$*.*

The algorithmic description of the SOI-KF is summarized in Algorithms 1 and 2. When dictated by the scheduling algorithm a sensor starts by running the predictor using (6) and (7) (step 1 in Algorithm 1) and whitens the observation $\mathbf{y}(n)$ (step 2). Subsequently, it recursively computes partial MMSE estimators via (16) and (18)-(20) in order to obtain the binary observations $b(n, m)$ by means of (17). When this process is complete, the message $\mathbf{b}(n)$ is transmitted. Algorithm 2 is continuously run by the sensors to estimate the state $\mathbf{x}(n)$. At each time slot n, *all* the sensors compute the predictors along with $\mathbf{H}_0(n)$ and move on to process the received message $\mathbf{b}(n)$. Processing $\mathbf{b}(n)$ entails recursive application of (18)-(20) for the M entries of $\mathbf{b}(n)$. The output of this process is the MMSE estimate $\hat{\mathbf{x}}(n|n)$.

4. PERFORMANCE ANALYSIS

Let us compare $\mathrm{tr}[\mathbf{M}(n|n)]$ and $\mathrm{tr}[\mathbf{M}(n|n-1)]$ for the SOI-KF with $\mathrm{tr}[\mathbf{M}^{\mathrm{K}}(n|n)]$ and $\mathrm{tr}[\mathbf{M}^{\mathrm{K}}(n|n-1)]$ reserved to denote the corresponding quantities for the clairvoyant KF which relies on the analog-amplitude observations $\mathbf{y}(n)$. Since the ECMs for the SOI-KF are independent of the data, we can find $\mathbf{M}(n|n)$ and $\mathbf{M}(n|n-1)$ by solving a discrete-time Ricatti equation [7]. A better insight, though, can be gained by recalling the underlying continuous-time model. We start with the following definition.

Definition 1 *Consider the continuous-time model* (1)-(2) *and a family of corresponding discrete-time models* (3) *parameterized by* T_s*. Let* $\mathbf{M}(T_s; n|n)$ *and* $\mathbf{M}(T_s; n|n-1)$ *be the ECM of the filtered and predicted estimates of the SOI-KF in Proposition 1 when sampling period* T_s *is used. Then, we define the continuous-time ECM as*

$$\mathbf{M}_c(t) := \mathbf{M}_c(nT_s) := \lim_{T_s \to 0} \mathbf{M}(T_s; n|n) = \lim_{T_s \to 0} \mathbf{M}(T_s; n|n-1).$$

An equivalent definition can be written for the clairvoyant KF whose continuous-time ECM will be denoted as $\mathbf{M}_c^{\mathrm{K}}(t)$ [4]. Another definition is that of the $(\pi/2)$-equivalent system:

Definition 2 *Consider a model as in* (1)-(2)*, with observation noise covariance* $\mathbf{C}_{v_c}(t, k)$*. We say that a model with otherwise identical parameters but noise covariance* $\mathbf{C}_{v_c}^{\pi/2}(t, k) = (\pi/2)\mathbf{C}_{v_c}(t, k)$*, is* $(\pi/2)$*-equivalent. For a given sampling period* T_s*, the KF for this latter model will be henceforth called the* $(\pi/2)$*-KF. We will denote its filtered and predicted ECM as* $\mathbf{M}^{\pi/2}(T_s; n|n)$ *and* $\mathbf{M}^{\pi/2}(T_s; n|n-1)$ *and the continuous-time ECM as* $\mathbf{M}_c^{\pi/2}(t)$*.*

Using Definitions 1 and 2, we can establish the relationship between the MSEs of the SOI-KF and the KF as follows.

Theorem 1 *For the state-observation model in* (1) – (2) *and its corresponding* $(\pi/2)$*-equivalent system, it holds that*

$$\mathbf{M}_c^{\pi/2}(t) = \mathbf{M}_c(t). \tag{21}$$

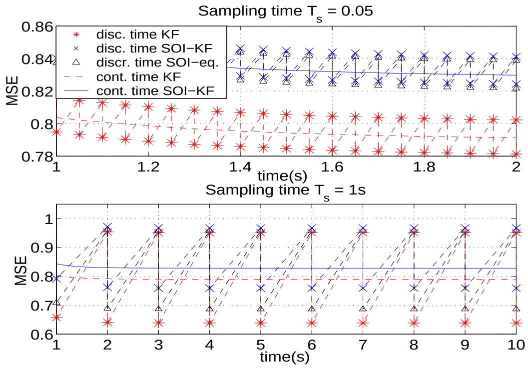

Fig. 2. MSE's for SOI-KF, KF and $(\pi/2)$-KF in discrete and continuous time ($\mathbf{A}_c(t) = \mathbf{I}$, $\mathbf{h}_c(t) = [1,\ 2]^T$, $\mathbf{C}_{u_c}(t) = \mathbf{I}$, and $\sigma^2_{v_c}(t) = 1$).

Theorem 1 establishes that the MSE of the SOI-KF is closely related to the MSE of the $(\pi/2)$-KF, since as $T_s \to 0$ the MSEs of these two filters are equal. Fig. 2 compares the KF, the SOI-KF and the $(\pi/2)$-KF for two T_s values. For large T_s, $\mathrm{tr}[\mathbf{M}^{\pi/2}(T_s; n|n)]$ and $\mathrm{tr}[\mathbf{M}(T_s; n|n)]$ are not equal (bottom); but as T_s decreases, these two quantities eventually coincide (top) since, as asserted by Theorem 1, $\mathrm{tr}[\mathbf{M}_c(nT_s)] = \mathrm{tr}[\mathbf{M}_c^{\pi/2}(nT_s)]$. We finally stress that the gap between the KF and the SOI-KF is small for small/moderate T_s.

5. TARGET TRACKING WITH SOI-EKF

Consider K sensors randomly and uniformly deployed in a square region of $2L \times 2L$ meters and suppose that sensor positions $\{\mathbf{x}^k\}_{k=1}^K$ are known. The WSN is deployed to track the position $\mathbf{x}(n) := [x_1(n), x_2(n)]^T$ of a target, whose state model accounts for $\mathbf{x}(n)$ and the velocity $\mathbf{v}(n) := [v_1(n), v_2(n)]^T$, but not for the acceleration that is modelled as a random quantity,

$$\begin{aligned} \mathbf{x}(n) &= \mathbf{x}(n-1) + T_s\mathbf{v}(n-1) + T_s^2/2\mathbf{u}(n) \\ \mathbf{v}(n) &= \mathbf{v}(n-1) + T_s\mathbf{u}(n), \end{aligned} \quad (22)$$

with T_s the sampling period and $p(\mathbf{u}(n)) = \mathcal{N}(\mathbf{u}(n); \mathbf{0}; \sigma_u^2\mathbf{I})$. Sensors gather information about their distance to the target by measuring the received power of a pilot signal following the path-loss model

$$y_k(n) = \alpha \log \|\mathbf{x}(n) - \mathbf{x}^k\| + v(n), \quad (23)$$

with $\alpha \geq 2$ a constant, $\|\mathbf{x}(n) - \mathbf{x}^k\|$ denoting the distance between the target and S_k, and $v(n)$ the observation noise with distribution $p(v(n)) = \mathcal{N}(v(n); 0; \sigma_v^2)$. Mimicking an extended (E)KF approach, we linearize (23) in a neighborhood of $\hat{\mathbf{x}}(n|n-1)$ to obtain

$$y_k(n) - y_k^0(n) \quad \approx \quad \mathbf{h}^T(n)\mathbf{x}(n) + v(n), \quad (24)$$

where $\mathbf{h}(n) := \alpha\hat{\mathbf{x}}(n|n-1)/\|\hat{\mathbf{x}}(n|n-1) - \mathbf{x}^k\|^2$ and $y_k^0(n)$ is an explicit function of α, $\hat{\mathbf{x}}(n|n-1)$ and $\mathbf{x}^k$.

The approximate model in (22)-(24) is of the form (3) and we can apply the SOI-KF outlined in Algorithms 1 and 2 to track the target's position $\mathbf{x}(n)$. This procedure amounts to the implementation of an extended SOI-(E)KF which is a low communication cost version of the EKF. The results of simulating this setup are depicted in Fig. 3, where we see that the SOI-KF succeeds in tracking the target with distance error for the position estimates of less than 10 meters (m). More important, note that the clairvoyant EKF and the SOI-EKF have almost identical performance even when the former relies on analog-amplitude observations and the SOI-EKF on the transmission of only a single bit per sensor.

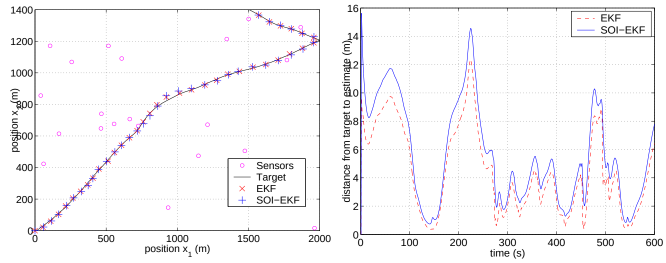

Fig. 3. Target tracking with EKF and SOI-EKF ($T_s = 1s$, $L = 2km$, $K = 100$, $\alpha = 3.4$ $\sigma_u = 0.2m$, $\sigma_v = 1$).

6. CONCLUDING REMARKS

Relying on the sign of innovations (SOI), we considered the problem of distributed state estimation in the context of wireless sensor networks. The binary SOI data destroys the linearity of the problem and leads to prohibitively complex MMSE state estimation. This motivated an approximation leading to the SOI-Kalman filter (KF) which offers an approximate MMSE estimator whose complexity and performance are very close to that of the clairvoyant KF even though the latter is based on the original (analog-amplitude) observations and the SOI-KF is based on the transmission of a single bit per observation. Relating the discrete-time KF and SOI-KF with the underlying continuous-time physical process monitored by the WSN, we established that the MSE of the SOI-KF coincides with the MSE of a KF applied to an otherwise equivalent system model with $\pi/2$ larger observation noise covariance matrix[2].

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