Analysis of Positioning Uncertainty in Reconfigurable Networks of Heterogeneous Mobile Robots

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***Abstract*— This paper studies the localization accuracy of a team of mobile robots that perform Cooperative Localization. We provide an analytical expression for the upper bound in positioning accuracy as a function of a weighted connectivity graph for the network of relative position measurements in the robot group. This network has a time-varying topology determined by the availability of relative position measurements between pairs of robots. The weights of the connectivity graph depend on (i) the odometric and orientation accuracy of each robot, and (ii) the accuracy of the robot tracker on each member of the team that measures its relative position with respect to other robots in the group. The theoretical results are validated by extensive simulations.**

I. Introduction

The topic of *Cooperative Localization* has recently attracted the interest of many researchers due to the greater versatility that robotic teams provide. The key element enabling this versatility is the sensor sharing that occurs between the robots. The communication link that exists between them, enables information from the sensors of each robot to "diffuse" in a sense to all the robots of the team. The additional information made available to the robots by measuring their relative position and/or orientation and communicating localization information throughout the group, allows them to achieve higher levels of localization accuracy ([1], [2]). External positioning information from a GPS receiver or a map of the environment can further increase the overall accuracy. In this work we primarily consider the most challenging scenario where the absolute positions of the robots cannot be measured or inferred. In this case the uncertainty in the position estimates for all robots will continuously increase. The case where absolute positioning information is available to at least one of the robots in the group, is subsumed in our formulation and is treated as a special case.

The theoretical analysis of the positioning uncertainty propagation during cooperative localization has been an open problem to this date. In [3] the theoretical treatment for determining upper bounds on the localization uncertainty for a homogeneous group of N robots by directly solving the continuous time Riccati equation for the covariance of the errors in the position estimates has been presented. In this work, the constraint that all robots measure the relative positions of all other robots in the group had been imposed. In the present work we lift this assumption and derive upper bounds for the uncertainty of arbitrary measurement topologies of the robotic team. Our motivation for this work stems from the fact that as the number of robots increases, the number of possible relative measurements increases as $O(N^2)$. Therefore the requirements on CPU time and communication bandwidth to transmit and process these measurements would make the scenario of processing relative measurements between all robots infeasible.

The availability of analytical expressions for the upper bounds for the localization uncertainty for any robotic team simplifies the task of designing a multirobot system. Task imposed constraints on the acceptable level of uncertainty for each robot can be employed to determine, during the design phase, the number of robots and accuracy of their sensors, and, once deployed, the measurement topology necessary to guarantee the required level of accuracy.

Throughout the paper we assume that all robots move randomly at the same time. Some (or all) of the robots of the team measure the relative positions of some other robots at each time step and use this information to improve the position estimates for all robots. A key element in this analysis is the Relative Position Measurement Graph (RPMG). This is defined as the graph whose vertices represent the robots of the group and its directed edges represent the measurements (Fig. 3). That is, if robot i measures the relative position of robot j, the RPMG contains a directed edge from vertex i to vertex j. The main result of this paper is that the rate of uncertainty increase in the group of robots is independent of the topology of the RPMG. However the connectivity of this graph affects the constant (time invariant) part of the covariance matrix that describes the localization uncertainty of the group, as well as, the time for the system to converge at steady state.

In the following section we outline the related existing approaches to cooperative localization. In Section III we present the formulation of the multi-robot localization problem and derive the Riccati equation that describes the time evolution of the maximum expected uncertainty in the position estimates. Sections IV and V describe the solution of the Riccati equation and derive the steady state localization uncertainty bounds for each of the cases considered. In Section VI simulation results are presented that validate the derived analytical expressions. Finally, in Section VII the conclusions of this work are drawn and future work directions are suggested.

II. Related Work

Many robotic applications require that robots work in collaboration in order to perform a certain task. When a group of robots needs to coordinate efficiently, precise localization is of critical importance. In these cases multi-robot cooperation for determining positioning estimates will result in improved localization accuracy by compensating for errors in odometry and/or a pose sensor.

Previous work on multiple robots has considered collaborative strategies when lack of landmarks made localization impossible. An example of a system designed for cooperative localization was first reported in [1]. A group of robots is divided into two teams in order to perform cooperative positioning. At each instant, one team is in motion while the other team remains stationary and acts as a landmark. The teams then exchange roles and the process continues until both teams have reached their target [4]. Similarly, in [5], only one robot moves, while the rest of the team of small-sized robots forms an equilateral triangle of localization beacons in order to update their pose estimates. Another implementation of cooperative localization is described in [6]. In this work a team of robots moves through the free space systematically mapping the environment. All previous approaches have the following limitations: (a) Only one robot (or team) is allowed to move at any given time, and (b) The two robots (or teams) must maintain visual (or sonar) contact at all times.

A different collaborative multirobot localization schema is presented in [7], [8]. The authors have extended the Monte Carlo localization algorithm [9] to the case of two robots when a map of the area is available to both of them. When these robots detect each other, the combination of their belief functions facilitates their global localization task. The main limitation of this approach is that it can be applied only within known indoor environments. In addition, since information interdependencies are being ignored every time the two robots meet, this method can lead to overly optimistic position estimates. This issue is discussed in detail in [10]. An approach that treats the issue of ignoring correlation terms, at the cost of increased computational requirements by introducing a dependency tree is presented in [11].

A Kalman filter based implementation of a cooperative navigation schema is described in [12]. In this case the effect of the orientation uncertainty in both the state propagation and the relative position measurements is ignored resulting in a simplified distributed algorithm. In [2], [13] a Kalman filter pose estimator is presented for a group of simultaneously moving robots. Each of the robots collects sensor data regarding its own motion and shares this information with the rest of the team during the update cycles. The Kalman filter is decomposed into a number of smaller communicating filters, one for every robot, processing sensor data collected by its host robot. It has been shown [10] that when every robot senses and communicates with its colleagues at all times, every member of the group has less uncertainty about its position than the robot with the best (single) localization results. Finally, in [14] and [15] an alternative to the Kalman filter approach is presented. A Maximum Likelihood estimator is used to process relative pose and odometric measurements recorded by the robots and a solution is derived by invoking numerical optimization.

To the best of our knowledge, there exist only few cases in the literature where analysis of the uncertainty propagation has been considered in the context of cooperative localization. In [12] the improvement in localization accuracy is computed after only a *single* update step with respect to the previous values of position uncertainty. In this case the robot orientations are assumed to be perfectly known and no expressions are derived for the propagation of the localization uncertainty with respect to time or the accuracy of the odometric and relative position measurements. In [16] the authors studied in simulation the effect of different robot tracker sensing modalities in the accuracy of cooperative localization. Statistical properties were derived from simulated results for groups of robots of increasing size N when only one robot moved at a time. In [3] a complete RPMG (Fig. 3(a)) and a homogeneous robot group is assumed and analytical expressions for the upper bounds of the localization uncertainty of the robots are derived.

We hereafter present the details of our method for deriving bounds for the localization uncertainty of a group of cooperating robots. Our problem formulation derives from that presented in our previous work [3], extended to the case of **arbitrary RPMGs** and **heterogeneous** groups of robots.

III. Problem Formulation

A group of N robots uses proprioceptive measurements (e.g., velocity) to propagate its state (position) estimates and exteroceptive measurements to update those estimates, using the Extended Kalman Filter (EKF). If the exteroceptive estimates only consist of relative position measurements, then from a Control Theoretic perspective the system is unobservable and the uncertainty of those estimates will monotonically increase. If additionally, the robots receive absolute positioning measurements (e.g., from a GPS receiver) the system becomes observable and at steady state the covariance of the state estimate will converge to a constant value. The basis of our approach for deriving bounds for the localization uncertainty is the use of the Riccati equation to describe the evolution of uncertainty over time. For a continuous time system, the Riccati equation that describes the propagation/update of the state estimates' covariance P is:

$$\dot{P} = FP + PF^T + G_c Q_c G_c^T - PH^T R^{-1} HP \qquad (1)$$

where F is the state transition matrix, the term $G_c Q_c G_c^T$ accounts for the covariance of the measurements used for propagation, and the term $H^T R^{-1} H$ represents the information input to the system by the measurements used for updates. For robots moving in 2-D, the kinematic equations are nonlinear and the matrices involved in the above equations are time varying. Thus we cannot solve directly for $P(t)$. However, we can obtain upper bounds for the uncertainty of the position estimates of the robots, if, instead of using these time-varying matrices in the solution of the Riccati, we employ their maximum expected values.

In our formulation, we assume that each robot is equipped with a sensor (such as a compass or a sun sensor) of limited accuracy that provides absolute orientation measurements.

This is required in the derivations that follow for determining bounds on the orientation uncertainty for each robot. If such a sensor is not available, our approach still holds under the condition that an upper bound for the orientation uncertainty is determined by alternative means, e.g. by estimating orientation from the structure of the environment around the robot [17], [18] or, by deriving an estimate for the maximum orientation uncertainty from odometry over a certain period of time for each robot [19].

We now describe the kinematic model we use for the robots. For a mobile robot moving on flat terrain, the continuous time kinematic equations are given by

$$x(k+1) = x(k) + V(k)\delta t\cos(\phi(k)) \quad (2)$$
$$y(k+1) = y(k) + V(k)\delta t\sin(\phi(k)) \quad (3)$$
$$\phi(k+1) = \phi(k) + \omega(k)\delta t \quad (4)$$

where $V(k)$ and $\omega(k)$ are the linear and rotational velocity of the robot at time k. Using measurements from the robot's proprioceptive sensors, we can write the following set of equations for propagating the estimate of the robot's pose:

$$\begin{aligned} \hat{x}(k+1|k) &= \hat{x}(k|k) + V_m(k)\delta t\cos(\hat{\phi}(k|k)) \\ \hat{y}(k+1|k) &= \hat{y}(k|k) + V_m(k)\delta t\sin(\hat{\phi}(k|k)) \\ \hat{\phi}(k+1|k) &= \hat{\phi}(k|k) + \omega_m(k)\delta t \end{aligned}$$

where

$$V_m(k) = V(k) - w_V(k), \qquad \omega_m(k) = \omega(k) - w_\omega(k)$$

are the measurements of the linear and rotational velocity of the robot respectively, contaminated by independent white zero-mean Gaussian noise processes with known variances:

$$\sigma_V^2 = E\{w_V^2\},\ \sigma_\omega^2 = E\{w_\omega^2\}$$

The robot also receives absolute orientation measurements

$$z(k+1) = \phi(k+1) + n_\phi(k+1)$$

with $n_\phi(k+1)$ a zero-mean white Gaussian noise process with known variance $\sigma_\phi^2 = E\{n_\phi^2\}$. These measurements are processed during the EKF update cycles.

A two-layer estimator for the robot's pose is employed. At the first layer of estimation, we use the odometric measurements of the rotational velocity, combined with the absolute orientation measurements, in order to estimate the robot's orientation. These estimates are then fed to a second estimator, that provides estimates for the robot's position. It is evident that this estimator is suboptimal, since correlation terms between orientation and position estimates are ignored, however, it facilitates the derivation of a closed form solution for the positioning uncertainty of the robots at steady state. [1]

[1]Due to space limitations many of the details of the following derivations have been omitted. The interested reader is referred to [20] for a thorough description of the intermediate steps.

A. Orientation Estimation

A Kalman filter is employed for estimating the orientation of each robot. In this case Eq. (1) is one dimensional, with $P = \sigma_{\phi_o}^2$ (scalar), $F = 0$, $G_c = -1$, $Q_c = \sigma_\omega^2$, $R = \sigma_\phi^2$, and $H = 1$. At steady state, the covariance of the orientation estimate will converge to a constant value. Using $\lim_{t\to\infty} \dot{P} = 0$, the Riccati can be written as

$$0 = \sigma_\omega^2 - \frac{1}{\sigma_\phi^2}(\sigma_{\phi_o}^2)^2 \Rightarrow \sigma_{\phi_o}^2 = \sigma_\phi\sigma_\omega$$

The last expression provides the steady state uncertainty in the estimate for the robot's orientation, $\sigma_{\phi_o}^2 = E\{(\phi - \hat{\phi})^2\}$.

B. Position propagation

The position of the robot is propagated using odometric measurements and the estimates for the robot's orientation, $\hat{\phi}$, provided by the first layer of estimation. By linearizing Eqs. (2), (3), the position error propagation equations for the robot can be written as

$$\begin{bmatrix} \widetilde{x}(k+1) \\ \widetilde{y}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}(k) \\ \widetilde{y}(k) \end{bmatrix} + $$
$$+ \begin{bmatrix} \delta t\cos(\hat{\phi}(k)) & -V_m(k)\delta t\sin(\hat{\phi}(k)) \\ \delta t\sin(\hat{\phi}(k)) & V_m(k)\delta t\cos(\hat{\phi}(k)) \end{bmatrix} \begin{bmatrix} w_V(k) \\ \widetilde{\phi}(k) \end{bmatrix}$$
$$\Leftrightarrow \widetilde{X}(k+1) = \Phi(k)\widetilde{X}(k) + G(k)W(k) \quad (5)$$

where $\Phi(k) = I$, is the 2×2 identity matrix,

$$Q_c(k) = E\{W(k)W^T(k)\} = \begin{bmatrix} \sigma_V^2 & 0 \\ 0 & \sigma_{\phi_o}^2 \end{bmatrix} \quad (6)$$

and we define

$$Q_d(k) = \frac{1}{\delta t} G(k)Q_c(k)G^T(k) \quad (7)$$

In this last relation, the scaling factor δt^{-1} ensures that the uncertainty influx in the system due to the odometric measurements is appropriately scaled with the sampling frequency of these measurements [21].

As evident from Eq. (7), the covariance $Q_d(k)$ of all sources of uncertainty and noise during propagation is a time-varying matrix. The values of the elements of this matrix depend on the measured velocity $V_m(k)$ of the robot and the estimate of its orientation $\hat{\phi}(k)$. In order to derive upper bounds for the covariance in the position estimates, we solve the Riccati equation, Eq. (1), using the expected value of $G_cQ_cG_c^T$. By averaging over all possible values of orientation and assuming a constant velocity for the robot, the mean value for the covariance in Eq. (7) is obtained:

$$\bar{Q}_d(k) = q\delta t I_{2\times 2},\ \bar{Q}_c(t) = qI_{2\times 2},\ \text{with } q = \frac{\sigma_V^2 + \sigma_{\phi_o}^2 V^2}{2} \quad (8)$$

where $I_{2\times 2}$ is the 2×2 identity matrix.

When no relative positioning information is available, the covariance for the position of the robot is propagated using only odometric information. This is described by the Riccati equation $\dot{P} = \bar{Q}_c = qI_{2\times 2}$, that yields

$$P(t) = P(0) + qtI_{2\times 2} \quad (9)$$

where $P(0)$ is the initial positioning uncertainty of the robot. As it is evident, the covariance (uncertainty) for the position of a single robot increases, on the average, linearly with time at a rate of q determined in Eq. (8) that depends on the accuracy of the absolute orientation measurements (σ_ϕ) and the robot's odometry (σ_V, σ_ω, V).

C. Relative Position Measurement Model

At this point instead of one robot, we consider the case of a group of robots where each of them (i) estimates its orientation by fusing rotational velocity measurements with absolute orientation measurements, (ii) propagates its position using the previous orientation estimates and linear velocity measurements, and (iii) measures the relative position z_{ij} of some (or all) other robots in the team:

$$z_{ij} = C^T(\phi_i)\left(\vec{p}_j - \vec{p}_i\right) + n_{z_{ij}} \tag{10}$$

where $\vec{p}_i$ is the position vector of the ith robot, i.e. $\vec{p}_i = [x_i\ y_i]^T$ expressed with respect to the global frame of reference, $C(\phi_i)$ is the rotational matrix representing the orientation of the observing robot, and $n_{z_{ij}}$ is the noise affecting the measurement, assumed to be white zero-mean Gaussian. By linearizing Eq. (10), the measurement error is obtained:

$$\begin{aligned} \widetilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1) \\ &= H_{ij}(k+1)\widetilde{X}(k+1) + \Gamma(k+1)n_{ij}(k+1) \end{aligned}$$

where

$$\begin{aligned} H_{ij}(k+1) &= C^T(\hat{\phi}_i(k+1))\ {}^oH_{ij} \\ {}^oH_{ij} &= \begin{bmatrix} \mathbf{0}_{2\times 2} & .. & \underbrace{-I_{2\times 2}}_{i} & .. & \underbrace{I_{2\times 2}}_{j} & .. & \mathbf{0}_{2\times 2} \end{bmatrix} \\ \widetilde{X}(k+1) &= \begin{bmatrix} \widetilde{p}_1 & \ldots & \widetilde{p}_i & \ldots & \widetilde{p}_j & \ldots & \widetilde{p}_N \end{bmatrix}^T \\ \Gamma(k+1) &= \begin{bmatrix} I_{2\times 2} & -C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij}(k+1) \end{bmatrix} \\ J &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad n_{ij}(k+1) = \begin{bmatrix} n_{z_{ij}}(k+1) \\ \widetilde{\phi}_i(k+1) \end{bmatrix} \\ \widehat{\Delta p}_{ij}(k+1) &= \hat{p}_j(k+1) - \hat{p}_i(k+1) \end{aligned}$$

The covariance for the measurement error is given by

$$\begin{aligned} R_{ij}(k+1) &= \Gamma(k+1)E\{n_{ij}(k+1)n_{ij}^T(k+1)\}\Gamma^T(k+1) \\ &= R_{z_{ij}}(k+1) + R_{\tilde{\phi}_{ij}}(k+1) \end{aligned} \tag{11}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_{ij}(k+1)$. More specifically, $R_{z_{ij}}(k+1)$ is the covariance of the noise $n_{z_{ij}}(k+1)$ in the recorded relative position measurement $z_{ij}(k+1)$ and $R_{\tilde{\phi}_{ij}}(k+1)$ is the additional covariance term due to the error $\widetilde{\phi}_{ij}(k+1)$ in the orientation estimate $\hat{\phi}_i(k+1)$ of the observing robot i. From Eq. (11), we can infer that this covariance matrix is a time-varying function of the position and orientation of the robots. In order to obtain an upper bound for the localization uncertainty, in [20] we derive the following expression for the maximum expected value of R_{ij}:

$${}^o\bar{R}_{ij} = \left({}^o\bar{r}_{z_{ij}} + {}^o\bar{r}_{\phi_i}\right) I_{2\times 2} \tag{12}$$

with ${}^o\bar{r}_{z_{ij}} = \frac{\sigma^2_{\rho_{ij}} + \rho_o^2\sigma^2_{\theta_{ij}}}{2}$ and ${}^o\bar{r}_{\phi_i} = \frac{\rho_o^2\sigma^2_{\phi_o i}}{2}$. It is assumed that the relative position measurement between two robots consists of a distance and a bearing measurement, whose errors are white zero-mean Gaussian and uncorrelated. In the last relation, $\sigma_{\rho_{ij}}$ and $\sigma_{\theta_{ij}}$ are the standard deviations of the distance and bearing measurements performed by robot i on robot j respectively, ρ_o is the maximum distance between any two robots in the group, and $\sigma^2_{\phi_o i}$ is the steady state uncertainty in the orientation estimate for robot i.

Note that due to the noise term attributed to the error in the orientation estimates of the measuring robot, the relative position measurements performed by one robot are *not* uncorrelated with eachother. In [20] it is shown that the maximum average value for the correlation term associated with the errors $\widetilde{z}_{ij}$ and $\widetilde{z}_{ik}$ is

$${}^o\bar{R}_{ijk} = \frac{1}{2}{}^o\bar{r}_{\phi_i} I_{2\times 2} \tag{13}$$

Employing the maximum average value for the covariance of the measurements, we compute the minumum avarage information available to the estimator:

$$H^T R^{-1} H = \Sigma_i\ H_i^T R_i^{-1} H_i = \Sigma_i\ {}^oH_i^T\ {}^o\bar{R}_i^{-1}\ {}^oH_i$$

where ${}^oH_i$ is a matrix whose block rows are ${}^oH_{ij}$, and $\bar{R}_i$ is the average covariance matrix for the measuremets performed by each robot. The diagonal block elements of this matrix are given by Eq. (12), while the off-diagonal block elements are given by Eq. (13).

Up to this point, only relative position measurements have been considered. If any of the robots, e.g., robot i, has access to absolute positioning information, such as GPS measurements or from a map of the area, the corresponding submatrix element of H is:

$$H_{i0} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \ldots & I_{2\times 2} & \ldots & \mathbf{0}_{2\times 2} \end{bmatrix}$$

while R_{i0} is provided by the absolute positioning sensor.

IV. Evaluation of the Riccati Equation

In this section we formulate the Riccati equation for the state covariance of the robot team and outline the steps that yield an analytical solution to it. The state vector for the entire robot team is defined as the stacked vector containing the position vectors of all N robots, i.e. a vector of dimension $2N$. Since the proprioceptive measurements of the N robots are uncorrelated, the matrix $G_c^T Q_c G_c$ for the Riccati equation is

$$G_c^T Q_c G_c = \mathbf{Diag}(\bar{Q}_{c_i}) = \mathbf{Diag}(q_i I_{2\times 2}) \tag{14}$$

where $\mathbf{Diag}(\bar{Q}_{c_i})$ is a block diagonal matrix, with elements the expected covariance matrices for the proprioceptive measurements of the robots, described by Eq. (8). For simplicity of notation, matrix $\mathbf{Diag}(\bar{Q}_{c_i})$ will be denoted as Q in the following.

From the kinematic model of the robots described by Eq. (5), the system propagation matrix, in continuous time, is $F = 0_{2N\times 2N}$. Substituting in Eq. (1), we have:

$$\dot{P} = Q - PH^T R^{-1} HP \tag{15}$$

For the solution of this matrix differential equation the standard methodology involving the decomposition of $P(t)$ into two matrices, and forming the Hamiltonian matrix is employed [22]. The main steps of the solution are described in what follows.

In order to facilitate the derivations, we define as P_n the *normalized covariance*:

$$P_n = Q^{-1/2} P Q^{-1/2}, \quad P = Q^{1/2} P_n Q^{1/2} \tag{16}$$

Substitution in Eq. (15) yields

$$\dot{P}_n = I_{2N\times 2N} - P_n C P_n \tag{17}$$

where $C = Q^{1/2} H^T R^{-1} H Q^{1/2}$. The solution is obtained by setting $P_n(t) = A_n(t) B_n^{-1}(t)$. Substitution in Eq. (17) results in a system of matrix differential equations, whose solution is

$$\begin{bmatrix} B_n(t) \\ A_n(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} B_n(0) \\ A_n(0) \end{bmatrix} \tag{18}$$

where the matrix $\mathcal{H}$ is the Hamiltonian of the system,

$$\mathcal{H} = \begin{bmatrix} \mathbf{0}_{2N\times 2N} & C \\ I_{2N\times 2N} & \mathbf{0}_{2N\times 2N} \end{bmatrix} \tag{19}$$

The initial values for $A_n(t)$ and $B_n(t)$ are selected so that the identity $P_n(0) = A_n(0) B_n^{-1}(0)$ holds, i.e. $A_n(0) = P_n(0)$ and $B_n(0) = I$. In order to derive an expression for the matrix exponential of Eq. (18), Taylor expansion of the exponential function, as well as the Singular Value Decomposition (SVD) of matrix C are employed. The SVD yields $C = U \Lambda U^T$ where U is an orthonormal matrix containing the singular vectors of C and Λ is a diagonal matrix whose diagonal elements are the singular values of C. By noting that $C = (R^{-1/2} H Q^{1/2})^T (R^{-1/2} H Q^{1/2})$, we can write $\Lambda = \mathrm{diag}(\lambda_i^2)$, where λ_i are the singular values of $(R^{-1/2} H Q^{1/2})$. Manipulation of each submatrix element of $e^{\mathcal{H}t}$ independently yields:

$$e^{\mathcal{H}t} = \frac{1}{2} \begin{bmatrix} U\,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T & U\,\mathrm{diag}\left(\lambda_i (e^{\lambda_i t} - e^{-\lambda_i t})\right) U^T \\ U\,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right) U^T & U\,\mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right) U^T \end{bmatrix}$$

At this point a comment regarding the eigenvalues of matrix C is due. It is shown in [20] that when at least one of the robots has access to absolute positioning information, matrix C is always nonsingular. In contrast, when all the robots in the team only record relative position measurements, this matrix is singular and has two eigenvalues equal to zero. Hence, in this case the notation used in the last expression presents a problem, since the eigenvalues appear in the denominator in the diagonal submatrix $e^{\mathcal{H}t}(2,1)$. Note that the quantities being divided by the zero eigenvalues are also equal to zero ($e^{0t} - e^{-0t} = 0$) and therefore the above quantity is actually undefined. However, in [20] it is proven formally that the quantity under consideration exists and is given by

$$e^{\mathcal{H}t}(2,1) = U \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{2\lambda_i}\right) & \mathbf{0}_{2\times(2N-2)} \\ \mathbf{0}_{(2N-2)\times 2} & t I_{2\times 2} \end{bmatrix} U^T$$

where diag_{2N-2} denotes a $(2N-2)\times(2N-2)$ diagonal submatrix. This expression is quite cumbersome and its use would make the resulting formulas unappealing and difficult to understand. We will therefore continue to use the initial, less strict notation in the following, bearing in mind that its true meaning is given by this last expression. Substitution of $e^{\mathcal{H}t}$ in Eq. (18) and subsequent substitution in $P_n(t) = A_n(t) B_n^{-1}(t)$ yields the closed form solution for the normalized localization uncertainty of the robots:

$$P_n(t) = U(K(t) + L(t)P_0)(L(t) + \Lambda K(t) P_0)^{-1} U^T \tag{20}$$

Where we have denoted

$$K(t) = \mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i}\right), \quad L(t) = \mathrm{diag}\left(e^{\lambda_i t} + e^{-\lambda_i t}\right)$$

and

$$P_0 = U^T P_n(0) U$$

V. Uncertainty Bounds at Steady State

The main results of this paper are presented in this section. The localization uncertainty of the robots at steady state is determined by computing the limit of Eq. (20) at steady state, i.e. after sufficient time. The special case of zero initial uncertainty is treated first:

1) Special Case - Zero Initial Covariance: When the uncertainty of the initial positions of the robots is zero, Eq. (20) reduces to

$$P_n(t) \quad = \quad U\,\mathrm{diag}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i (e^{\lambda_i t} + e^{-\lambda_i t})}\right) U^T \tag{21}$$

The limit of this quantity as $t \to \infty$ depends on the eigenvalues λ_i of matrix C and thus on the type of measurements that the robots receive.

(a) When at least one of the robots receives absolute position measurements, such as GPS, all eigenvalues of C are positive and the uncertainty converges to a constant value at steady state. By simple calculation of the limit $\lim_{t\to\infty} P_n(t)$ in Eq. (21) and substitution of this result for the normalized covariance in $P = Q^{1/2} P_n Q^{1/2}$, the maximum expected localization uncertainty of the robots at steady state is found to be:

$$P_{ss}(t) = Q^{1/2} U\,\mathrm{diag}\left(\frac{1}{\lambda_i}\right) U^T Q^{1/2} = Q^{1/2} \sqrt{C^{-1}} Q^{1/2} \tag{22}$$

where $\sqrt{C^{-1}} = U \Lambda^{-1/2} U^T$ is the matrix square root of C^{-1}.

(b) When none of the robots has access to absolute position measurements, there exist two eigenvalues equal to zero and the *normalized covariance* can be written as:

$$P_n(t) = U \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{e^{\lambda_i t} - e^{-\lambda_i t}}{\lambda_i (e^{\lambda_i t} + e^{-\lambda_i t})}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & t\ I_{2\times 2} \end{bmatrix} U^T$$

The maximum expected uncertainty at steady state is given by

$$\begin{aligned} P(t) \quad &= \quad Q^{1/2} U \begin{bmatrix} \mathrm{diag}_{2N-2}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} U^T Q^{1/2} \\ &+ \quad t\ \left(Q^{1/2} U_{2N-1} U_{2N-1}^T Q^{1/2} + Q^{1/2} U_{2N} U_{2N}^T Q^{1/2}\right) \end{aligned}$$

where U_{2N-1} and U_{2N} are the two eigenvectors of C corresponding to the zero eigenvalues. It is shown in [20] that the quantity $Q^{1/2} U_{2N-1} U_{2N-1}^T Q^{1/2} + Q^{1/2} U_{2N} U_{2N}^T Q^{1/2}$ is *independent* of the topology of the RPMG. Also, it is straightforward to verify that the first term of the above equation converges to a constant term as the system approaches steady state. These observations lead to the following lemma:

Lemma 1: For a heterogeneous team of N mobile robots performing Cooperative Localization, if the initial covariance of their position estimates is zero, the rate of increase of the maximum expected localization uncertainty of the robots at steady state is

$$\dot{P}_{ss}(i,i) = q_T t$$

where $\frac{1}{q_T} = \sum_{i=1}^{N} \frac{1}{q_i}$ and q_i is defined for each robot by Eq. (8). Thus the rate of increase of uncertainty for *all* the robots is equal to q_T and *independent* of the topology of the RPMG, as long as the graph is connected.

We next present the result for the case of nonzero initial covariance and defer further discussion of this important result for Section VI.

2) General Case - Nonzero Initial Covariance: It is often the case that the initial knowledge of the position of the robots is imprecise. In such cases the initial covariance matrix is assumed to be an arbitrary positive semi-definite matrix and the normalized uncertainty is shown in [20] to be equal to

$$P_n(t) = U(KL^{-1} + 4L^{-1}P_0(I + \Lambda K L^{-1} P_0)^{-1} L^{-1})U^T \quad (23)$$

where time arguments have been dropped for simplicity of notation.

(a) If at least one of the robots receives absolute position measurements, all the eigenvalues of C are positive and as shown in [20] the steady state uncertainty is:

$$P_{ss}(t) = Q^{1/2} U \operatorname{diag}\left(\frac{1}{\lambda_i}\right) U^T Q^{1/2} = Q^{1/2}\sqrt{C^{-1}}Q^{1/2}$$

This result is identical with the result for the special case of $P(0) = 0$ (Section V-1a). Notice that at steady state the uncertainty depends on the topology of the RPMG (affecting C) and the covariance of the proprioceptive and exteroceptive measurements, represented by Q and R (which is "embedded" in C).

(b) When none of the robots receives absolute position measurements, it is shown in [20] that there exist two singular values of C equal to zero. In this case the derivation of the final expression for the steady state uncertainty is quite lengthy and cannot be included here, due to limited space. The interested reader is referred to [20] for the details of the proof.

Lemma 2: The maximum expected steady state localization uncertainty of a group of mobile robots performing cooperative localization is given by:

$$P_{ss}(t) = Q^{1/2} U \begin{bmatrix} \operatorname{diag}_{2N-2}\left(\frac{1}{\lambda_i}\right) & \mathbf{0}_{(2N-2)\times 2} \\ \mathbf{0}_{2\times(2N-2)} & \mathbf{0}_{2\times 2} \end{bmatrix} U^T Q^{1/2}$$
$$+ q_T \begin{bmatrix} \alpha & \beta & \alpha & \beta & \cdots \\ \gamma & \delta & \gamma & \delta & \cdots \\ \alpha & \beta & \alpha & \beta & \cdots \\ \gamma & \delta & \gamma & \delta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + t\, q_T \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (24)$$

where $\frac{1}{q_T} = \sum_{i=1}^{N} \frac{1}{q_i}$ and the parameters α, β, γ, δ are defined as follows: Let $W = q_T Q^{-1} P(0)(I_{2N\times 2N} + Q^{-1/2}\sqrt{C}Q^{-1/2}P(0))^{-1}Q^{-1}$. Then $\alpha = \sum_{i,j \text{ odd}} w_{ij}$ will be the sum of all elements of $W = [w_{ij}]$ whose indices are both odd, $\delta = \sum_{i,j \text{ even}} w_{ij}$ will be the sum of all elements with two even indices, and $\gamma = \sum_{i \text{ odd}, \text{j even}} w_{ij}$ will be sum of all elements with an odd row index and an even column index. Due to symmetry, $\beta = \gamma$.

The first term of the above equation is a constant term, whose value depends on the topology of the RPMG and the characteristics of the sensors of the robots. The second term is a constant term depending on the initial uncertainty, as well as the characteristics of the robots and the RPMG topology. Finally, the last term contributes with a *constant rate* of uncertainty increase that is proportional to q_T. At this point we should note that the rate of uncertainty increase is *independent* of the initial uncertainty $P(0)$, the accuracy of the relative position measurements, and the topology of the RPMG. From the definition of q_T, it becomes clear that it will be smaller than the smallest of the q_i's (notice that the definition of q_T is analogous to the expression for the total resistance of resistors in parallel). This implies that it suffices to equip only *one* robot in the team with proprioceptive sensors of high accuracy, in order to achieve a desired rate of uncertainty increase. All the robots of the group will experience a reduction in the rate at which their uncertainty increases and this improvement is more significant for robots with sensors of poor quality. We further discuss the significance of Eq. (24) in the next section, where the results of our simulations are presented.

VI. Simulation Results

A series of experiments in simulation were conducted, with the aim of validating the preceding theoretical analysis. Robotic teams of different sizes and several topologies of the RPMG are considered and the covariance values predicted by our theoretical analysis are compared to the experimental results. For the simulations the same two-layer approach to the estimation of the robot's pose is employed that was used in the derivation of the theoretical bounds. For our experiments, the robots are restricted to move in an area of radius $r = 20$m, thus the maximum allowable distance between any two robots is $\rho_o = 40$m. The velocity of all robots is assumed to be constant, equal to $V_i = 0.25$m/sec. Note, however, that our analysis does *not* require all the robots to move at the same speed. The orientation of the robots, while they move, changes randomly using samples drawn from a uniform distribution.

The parameters of the noise that corrupts the proprioceptive measurements of the simulated robots are identical to those measured on a iRobot PackBot robot ($\sigma_V = 0.0125$m/sec, $\sigma_\omega = 0.0384$rad/sec). The absolute orientation of each robot was measured by a simulated compass with $\sigma_\phi = 0.0524$rad. The robot tracker sensor returned range and bearing measurements corrupted by zero-mean white Gaussian noise with $\sigma_\rho = 0.01$m and $\sigma_\theta = 0.0349$rad. The above values are compatible with noise parameters observed in laboratory experiments [23]. All measurements were available at 1Hz.

In order to demonstrate the validity of the derived formulas for the steady state localization uncertainty of the robots, in Fig. 1 we plot the true value vs. the theoretical bound for the covariance along the x-axis of a robot performing cooperative localization. For this specific experiment, a team of 2 robots was simulated and the parameters for the proprioceptive sensors of the robots were chosen so that the second robot has 5 times less accurate measurements compared to the first one (i.e, for this robot $\sigma_{V_2} = 0.0625$m/sec, $\sigma_{\omega_2} = 0.192$rad/sec). As evident, the true covariance consistently remains below the maximum expected value predicted. This behavior for the localization uncertainty is a typical example of the results of our simulation experiments.

At this point we focus our attention on the effect of the network topology on localization accuracy. In order to preserve the clarity of the figures, we hereafter consider a homogeneous team of robots (i.e. a team whose robots are equipped with sensors of equal accuracy). Note however, that homogeneity is not a prerequisite of our approach, as Fig. 1 demonstrates.

In Fig. 2 the localization uncertainty evolution is presented for a team of 9 robots with changing RPMG topology. Initially up to $t = 200$sec, the robots do not record any relative position measurements and propagate their position estimates using Dead Reckoning (DR). At $t = 200$sec the robots start receiving relative position measurements and the topology of the RPMG is a complete one (Fig. 3(a)). The significant improvement in the rate of uncertainty increase that is achieved by using relative positioning information is demonstrated in this transition. At $t = 400$sec the RPMG assumes a ring topology (Fig. 3(b)). We note that the uncertainty undergoes a transient phase, during which it increases at a higher rate and then, once steady state is reached, the rate of increase is *identical* to the rate associated with the complete graph. This validates the result of Eq. (24) and shows that the dominant factor in determining the rate of localization uncertainty increase is the quality of the proprioceptive sensors of the robots.

At $t = 600$sec a supposed failure of the communication network occurs, and in the time interval between 600sec and 800sec only two robots are able to measure their relative position, (Fig. 3(c)). This case can be viewed as a degenerate case, where the 7 robots localize based solely on Dead Reckoning, while the other two robots form a smaller team. We can observe that the rate of increase of the covariance is larger when the team consists of only 2 robots, instead of 9.

At $t = 800$sec the RPMG assumes a non-canonical topology, i.e., random graph (Fig. 3(d)). This case is perhaps the most important for real applications, since robots will usually measure the distances of their neighbors and due to the robots' motion, the topology of the RPMG can change randomly. In this case, the uncertainty increases at a rate identical to that of cases I and II of the graph's topology, as predicted by our theoretical analysis. It is also apparent that the uncertainty for each robot converges to a set of lines with the same slope (rate of uncertainty increase), but different constant offset. This is due to the effect of the different degree of connectivity in the RPMG of each robot. Connection-rich robots have direct access to positioning information from more robots and thus attain lower positioning uncertainty.

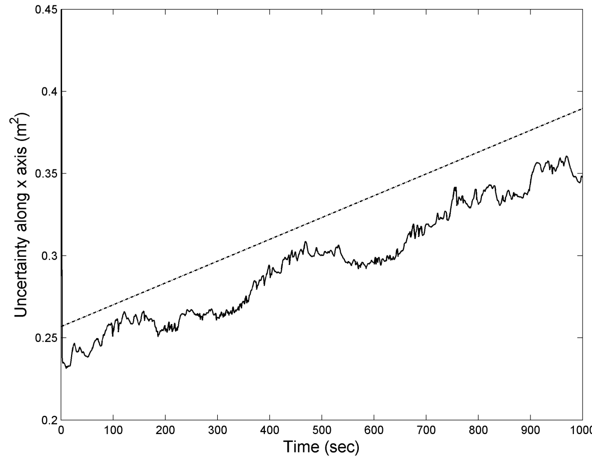


Fig. 1. True covariance vs. theoretical bound for a heterogeneous team of 2 robots.

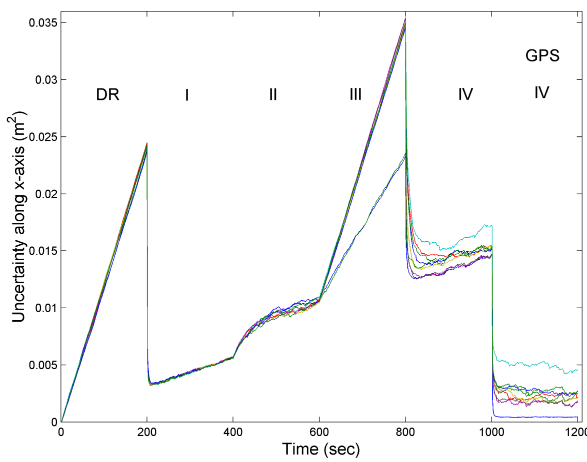
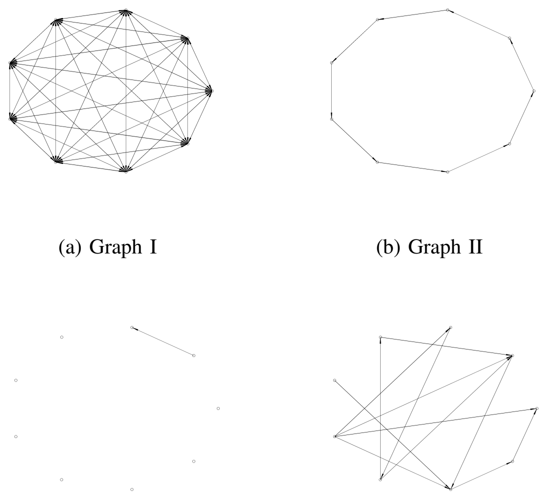


Fig. 2. Uncertainty evolution for a RPMG with changing topology.

At $t = 1000$sec only one of the robots starts receiving GPS measurements while the RPMG retains the topology of (Fig. 3(d)) The GPS measurements are corrupted by noise with a standard deviation of $\sigma_{GPS} = 0.05$m in each axis. It is evident that the availability of absolute position measurements to *any* robot drastically reduces the localization uncertainty for *all* the robots in the group. Furthermore, the system becomes observable and the uncertainty is bounded for all robots in the group. As in the previous case, the constant value to which the uncertainty for each robot converges, depends on its degree of connectivity.

VII. Conclusions

The problem of positioning uncertainty build-up for heterogeneous robotic teams of arbitrary and potentially dynamic relative measurement topologies has been studied. We have derived closed form expressions for the maximum expected



(a) Graph I (b) Graph II

(c) Graph III (d) Graph IV

Fig. 3. The four different measurement graph topologies considered in the simulations. Each arrow represents a relative position measurement, with the robot (node) where the arrow starts being the observing robot.

localization uncertainty that can be employed, early on during the design phase, for determining the positioning capabilities of a multi-robot system. For a robot team whose members only register relative position measurements, Lemma 2 maintains that the rate of uncertainty growth at steady state is *independent* of both the accuracy of the robot tracker device and the topology of the Relative Position Measurement Graph (RPMG). Besides the number of robots comprising the team, the single most important factor that determines the uncertainty of position estimates is the accuracy of the proprioceptive and orientation sensors of the robots. In the particular case of a heterogeneous robot group, the accuracy of the *best* equipped robot is the one that has the greatest impact on overall accuracy. These conclusions are of great practical importance for implementation purposes, since they ensure that the use of a minimum number of relative position measurements, when computational and communication resources are limited, only inflicts a small penalty on localization performance (constant offset), while sustaining the same rate of uncertainty increase. Finally, when any of the robots has access to absolute positioning measurements, such as GPS, the positioning uncertainty of *all* the robots in the group remains bounded and converges to a constant value.

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