

3D Relative Pose Estimation from Distance-Only Measurements

Nikolas Trawny*, Xun S. Zhou*, Ke X. Zhou**, and Stergios I. Roumeliotis*[†]

University of Minnesota, Minneapolis, MN 55455

Email: {trawny|zhou|kezhou|stergios}@cs.umn.edu

Abstract—In this paper, we develop an algorithm for determining the relative position and attitude of two robots moving in 3D, using only dead-reckoning and inter-robot distance measurements. From the mechanical analogue of parallel manipulators, it is known that this problem has 40 solutions when six distance measurements are available. These general solutions are not known in closed-form, and existing closed-form solutions require additional bearing sensors, or impose strong constraints on the geometric structure of the manipulator (in our case, the robots’ trajectories). This paper presents, for the first time, an efficient, algebraic algorithm to solve the relative pose using 10 distance measurements, without imposing any constraints on the robots’ motion. We further present a weighted least-squares refinement step, and validate our algorithm in various simulations, demonstrating its efficiency and accuracy.

I. INTRODUCTION

Using multiple robots to cooperatively accomplish tasks can lead to significant performance improvement compared to single-robot deployment. One of the premises for successful cooperation is accurate navigation, in particular precise relative localization with respect to the fellow robots.

The predominant sensors used for cooperative localization are laser scanners and cameras, which can provide range and bearing or bearing-only information. Range-only sensors, e.g., based on sound or radio frequency waves, have received comparatively little attention, despite having a number of advantages. In particular, they are insensitive to changes in lighting conditions, can penetrate certain obstacles depending on the wavelength (and thus do not require direct line of sight), and allow for easy data association by embedding a signature in the transmitted signal. For these reasons, range-only sensors have found widespread use in underwater or underground scenarios.

This paper addresses the problem of determining the relative pose (i.e., position and orientation) between two robots moving in 3D, using only dead-reckoning (e.g., by means of an inertial measurement unit (IMU)) and robot-to-robot distance measurements. Imagine two robots meet for the first time, both having estimated their pose with respect to their own global frame of reference. In order to exchange and fuse data, such as an acquired map of the area, both robots need to know the relative transformation between their global frames. To this end, the two robots will measure the distance between them from several different locations. Our

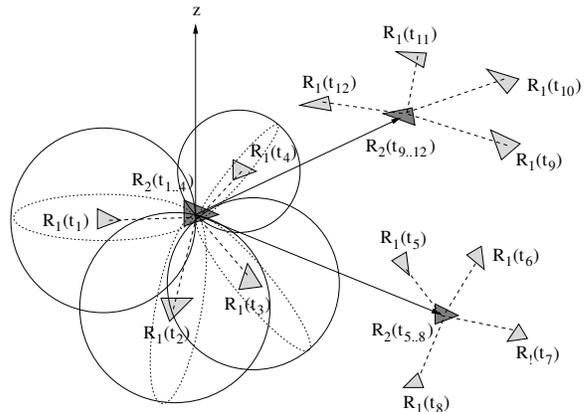


Fig. 1. A simplistic approach to determine the robot-to-robot relative transformation using 12 distance measurements. Robot R_1 determines robot R_2 ’s position at three different locations, using triangulation of four distance measurements each. The light (dark) triangles denote the location of robot R_1 (R_2), and $t_i, t_{i..j}, i, j \in \{1, \dots, 12\}$ indicate the time step(s) that a robot remains at a certain location.

goal is to determine the relative pose using these distance measurements and the robots’ estimated positions at the time the measurements are taken.

A simplistic solution, as shown in Fig. 1, is based on a *coordinated*, specific sequence of motions. If robot R_1 can measure the distance to robot R_2 from four different positions, while the latter is stationary, robot R_1 can uniquely triangulate robot R_2 ’s position with respect to its own frame of reference. Repeating this process for two other, (non-collinear) positions of robot R_2 , its orientation with respect to R_1 can be determined. In total, this process requires 12 measurements, and, what is worse, imposes strong constraints on the robot trajectories.

In this paper, we develop an efficient, algebraic method to determine the relative pose, using *only 10 distance measurements*, and imposing *no constraints* on the robot trajectories. To our knowledge, the problem of pose reconstruction from distance-only measurements, in particular for general geometries, has not yet been addressed in the literature.

However, a solution to this problem has many potential applications. One such area is tight integration and robustification of GPS-based localization [1]. In particular, our algorithm could allow localization in the presence of fewer than the usually required four GPS satellites, or in remote space applications where no dedicated GPS infrastructure is available. On Mars, for example, range measurements to satellites in orbit could be used for planetary rover localiza-

*Dept. of Computer Science & Engineering

**Dept. of Electrical Engineering

[†]This work was supported by the University of Minnesota (DTC), the NASA Mars Technology Program (MTP-1263201), and the National Science Foundation (EIA-0324864, IIS-0643680).

tion. Another wide field of application is in navigation of multi-robot teams, in particular for aerial, underground, or underwater applications.

Finally, it should be noted that this problem has an interesting mechanical analogue, namely the forward kinematics of a certain class of parallel manipulators, the general Stewart-Gough Platforms [2]. Hence, apart from mobile robotics, the algorithm presented in this paper is also directly applicable to problems in robotic manipulators, which have been the subject of intensive research during the last 20 years.

After a short review of related literature (cf. Section II), we provide a mathematical problem formulation in Section III. Section IV presents the uniqueness argument when using 10 measurements. In Sections V and VI, we describe our algebraic method, together with a Weighted Least-Squares (WLS) refinement scheme. Finally, in Section VII we show simulation results to demonstrate our algorithm's performance in the presence of noise, followed by concluding remarks in Section VIII.

II. LITERATURE REVIEW

Most applications of localization using range sensors concern the case of position-only estimation based on triangulation, i.e., the intersection of several concurrent range measurements to beacons at known positions. Examples are GPS [1], and localization using distance measurements by means of signal strength or time of flight in sensor networks [3], [4], [5]. Additionally, in the mobile robotics research, range sensors have been employed in single-robot localization and mapping [6], [7], [8].

Work more closely related to relative pose determination in 3D is presented by Higo *et al.* [9], who use range measurements between a stationary robot on the surface of an asteroid and an orbiting satellite to determine the position of the robot. Since the robot remains stationary and the orbital parameters of the satellite and the turn rate of the asteroid are considered known, the problem can be solved via triangulation.

Beutler and Hanebeck [10] solve the relative pose problem for two moving objects in 3D, however, they assume that multiple *simultaneous* distance measurements from any one point on the first body to many points on the other are available. That way, the location of each sensor point can be triangulated separately, which greatly simplifies the problem.

A number of interesting results pertinent to the relative pose problem have been published in the domain of robot manipulators. A general Stewart-Gough platform [2] represents the mechanical analogue to our problem. Such a platform consists of six articulated legs of variable length, attached to a fixed base and a moving platform at six arbitrary but fixed points through universal and spherical joints. The *forward kinematics problem* consists of determining the position and orientation of the platform relative to the base, given the length of the legs, and the coordinates of the attachment points in their respective frame of reference (base or platform). Alternatively, this setting can also be viewed as an *assembly problem*, asking in what configurations platform,

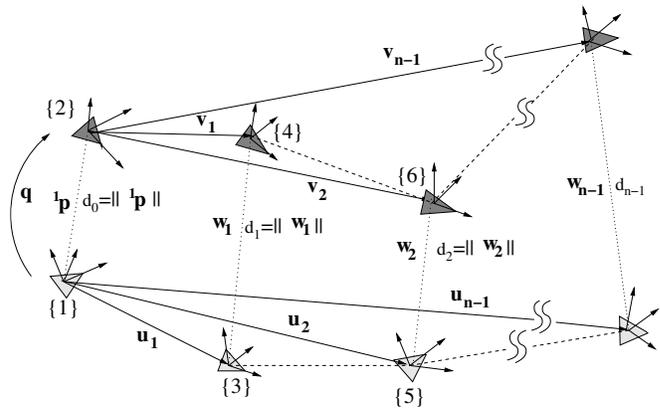


Fig. 2. The trajectories of robots R_1 and R_2 . The odd (even) numbered frames of reference depict the consecutive poses of robot R_1 (R_2). d_i , $i = 0, \dots, n-1$, denotes the distance between the two robots when at positions \mathbf{u}_i and \mathbf{v}_i , respectively.

base, and legs of given length can be assembled. Algebraic geometry has shown that the problem using the lengths of six legs has 40 solutions [11], all of which can be real [12]. Actual computation of these 40 solutions is quite involved [13], requiring to solve multivariate polynomial systems, for which (to the knowledge of the authors) no closed-form solution is known to exist. Several researchers have examined the use of constrained geometry and additional sensors in order to uniquely determine the true pose, and at the same time to simplify the solution. One example is the work of Bonev *et al.* [14], who use three additional distance measurements from passive legs, together with the assumption of a planar platform, which leads to a closed-form solution for the relative pose.

The problem of determining the relative pose between two robots from distance-only measurements has been solved for the case of motion in 2D [15]. In this paper we extend the solution to 3D. To the best of our knowledge, this paper is the first to present a closed-form solution to the relative pose problem, using only 10 distance measurements and requiring no restrictions on the robots' trajectories. In addition, our algorithm can be directly applied to solve the forward kinematics of a general Stewart-Gough platform. To elucidate our approach, we hereafter present the mathematical problem formulation.

III. PROBLEM FORMULATION

Consider two robots, R_1 and R_2 , moving randomly in 3D space (cf. Fig. 2). At the time when the robots meet for the first time, let the frames of reference attached to the robots coincide with their global frames and denote them by $\{1\}$ and $\{2\}$. Here, the robots record the first measurement of their distance, d_0 .

Further along their trajectories, the robots reach positions \mathbf{u}_i and \mathbf{v}_i , $i = 1, \dots, n-1$, which they can each estimate with respect to their own global frame, $\{1\}$ or $\{2\}$, respectively (e.g., from integrating IMU measurements over time). At this point, the robots record an additional measurement of their distance, d_i , which corresponds to the magnitude of

vector \mathbf{w}_i from robot R_1 to R_2 . In this fashion, the robots will acquire a total of n measurements.

Our goal is to determine the 6 degrees of freedom (d.o.f.) relative pose between robot R_1 and R_2 using the n distance measurements and the corresponding position estimates. Let the vector ${}^1\mathbf{p}$ denote the location of the origin of frame $\{2\}$ expressed with respect to frame $\{1\}$ and the unit quaternion \mathbf{q} their relative orientation

$$\begin{aligned} {}^1\mathbf{p} &:= [x_1 \ y_1 \ z_1]^T \\ \mathbf{q} &:= [q_1 \ q_2 \ q_3 \ q_4]^T := [\vec{\mathbf{q}}^T \ q_4]^T, \quad \mathbf{q}^T \mathbf{q} = 1 \end{aligned} \quad (1)$$

Using the (non-Hamiltonian) conventions for quaternions specified in [16], the orientation of frame $\{2\}$ relative to frame $\{1\}$ can be expressed by¹ a rotation matrix C whose entries are quadratic in the elements of \mathbf{q}

$$C = (2q_4^2 - 1)\mathbf{I}_{3 \times 3} - 2q_4[\vec{\mathbf{q}} \times] + 2\vec{\mathbf{q}}\vec{\mathbf{q}}^T \quad (2)$$

From the first distance measurement d_0 , we know that the position of robot R_2 lies on a sphere that satisfies the following equation

$$d_0^2 = {}^1\mathbf{p}^T {}^1\mathbf{p} \quad (3)$$

Generally, the inter-robot distance is given by the 2-norm of vector \mathbf{w}_i connecting the positions of robots R_1 and R_2 (cf. Fig. 2), i.e.

$$d_i = \|\mathbf{w}_i\|_2, \quad i = 1, \dots, n-1 \quad (4)$$

where

$$\mathbf{w}_i = {}^1\mathbf{p} + C\mathbf{v}_i - \mathbf{u}_i \quad (5)$$

Rearranging the terms in Eq. (4) after squaring both sides yields the following constraints

$$\begin{aligned} d_i^2 &= ({}^1\mathbf{p} + C\mathbf{v}_i - \mathbf{u}_i)^T ({}^1\mathbf{p} + C\mathbf{v}_i - \mathbf{u}_i) \\ \Rightarrow \varepsilon_i &= (\mathbf{u}_i - {}^1\mathbf{p})^T C\mathbf{v}_i + \mathbf{u}_i^T {}^1\mathbf{p}, \quad i = 1, \dots, n-1 \end{aligned} \quad (6)$$

where, for brevity,

$$\varepsilon_i := \frac{1}{2}(d_0^2 + \mathbf{u}_i^T \mathbf{u}_i + \mathbf{v}_i^T \mathbf{v}_i - d_i^2)$$

Notice that the unknown variables are embedded in ${}^1\mathbf{p}$ and C , whereas all quantities in ε_i are known (measured or estimated). We combine all constraints (1), (3), and (6) as

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \quad (7)$$

$$x_1^2 + y_1^2 + z_1^2 = d_0^2 \quad (8)$$

$$(\mathbf{u}_i - {}^1\mathbf{p})^T C\mathbf{v}_i + \mathbf{u}_i^T {}^1\mathbf{p} = \varepsilon_i, \quad i = 1, \dots, n-1 \quad (9)$$

From research in parallel manipulators [11] it is known that, given six distance measurements, this polynomial system has 40 solutions. As no closed-form solution is known yet, special algorithms based on homotopy continuation, such as PHCpack [17], are among the fastest numerical solvers used for this problem. Still, in our experiments, PHCpack

¹ $[\mathbf{a} \times]$ denotes the skew-symmetric 3×3 matrix corresponding to the cross-product, that is $[\mathbf{a} \times] \mathbf{b} = \mathbf{a} \times \mathbf{b}$.

required 25 seconds to solve one instance of this problem, a prohibitive delay for many robotic applications. Further difficulties arise in the presence of noise, or when more than six measurements become available, since even slight perturbations in the data can cause this system to have no solution.

The few cases reported in literature, for which closed-form solutions to this problem exist (e.g., [14]) require strong additional constraints to be imposed on the geometry of the system (i.e., the trajectories of the robots), or the use of additional sensors, which again is undesirable in mobile robotics.

In what follows, we will present an efficient algorithm that, using 10 distance measurements, can determine the unique solution in closed form, if it exists. To this end, we first provide an argument that 10 measurements will actually suffice to determine a unique solution.

IV. UNIQUENESS ARGUMENT

We know that given six distance measurements, the square polynomial system (7)-(9) has 40 solutions. Finding the unique solution that corresponds to the true relative pose requires additional measurements. This process adds extra constraints in the form of Eq. (9) to the polynomial system, which eliminate the spurious solutions obtained from the square system. The question that needs to be answered is how many additional measurements are required to eliminate all but the true solution. We will argue, that for generic robot trajectories, in general only one additional measurement suffices to identify the unique solution.

Assume we have solved the square system with six distance measurements, and have found 40 solutions denoted as \mathbf{x}_i , $i = 1, \dots, 40$. Without loss of generality, we assume \mathbf{x}_1 is the true solution. By taking an additional distance measurement, we will have an extra equation of the form of Eq. (9) that contains the quantities \mathbf{u}_6 , \mathbf{v}_6 , and d_6 . For simplicity we denote these parameters as a vector $\boldsymbol{\theta}$. The measured or estimated quantities $\boldsymbol{\theta}$ and the true solution \mathbf{x}_1 satisfy Eq. (9), which we write in compact form as $f(\boldsymbol{\theta}, \mathbf{x}_1) = 0$. We denote the set of all $\boldsymbol{\theta}$ that satisfy Eq. (9) as S_1 .

Next we show that when the two robots move to two new locations, in general only the true solution satisfies $f(\boldsymbol{\theta}, \mathbf{x}_1) = 0$. In other words, $f(\boldsymbol{\theta}, \mathbf{x}_i) \neq 0$, $i = 2, \dots, 40$ with probability one.

Note that there exist values of the parameters $\boldsymbol{\theta}$ that fulfill more than one solution \mathbf{x}_1 . However in such case, the two robots must move to specific positions under additional constraints. If for example, two solutions exist, then $\boldsymbol{\theta}$ needs to satisfy the following equations.

$$\begin{aligned} f(\boldsymbol{\theta}, \mathbf{x}_1) &= 0 \\ f(\boldsymbol{\theta}, \mathbf{x}_i) &= 0, \quad i \in \{2, \dots, 40\} \end{aligned}$$

We denote the set of solutions for $\boldsymbol{\theta}$ as S_2 . Obviously, the space of S_2 is constrained by two equations, while S_1 is only constrained by one equation. Thus $S_2 \subset S_1$ and S_2 is one dimension smaller than S_1 (more details can be

found in Chapter 9 of [18]). Therefore the probability of obtaining parameters θ satisfying two solutions is $|S_2|/|S_1|$ which is almost zero². Those sets of θ that fulfill more than two solutions are of even smaller dimension, and hence negligible.

Therefore, we conclude that with probability one, for generic robot trajectories, seven distance measurements are sufficient to obtain a unique solution.

V. ALGEBRAIC METHOD

We hereafter present an algebraic method for determining the unique relative pose between the two robots from at least 10 distance measurements. The key idea behind this approach is to introduce the new variables ${}^2\mathbf{p} := C^{T1}\mathbf{p}$, which allows us to reduce the order of the polynomial system from cubic to quadratic. The resulting system can then be solved by applying techniques similar to the ones used in the computer vision literature for solving the relative pose problem for two cameras from point or line correspondences [19], [20]. Using the new variables ${}^2\mathbf{p}$, we can rewrite Eq. (9) as

$$\mathbf{u}_i^T C \mathbf{v}_i + \mathbf{u}_i^T \mathbf{p} - \mathbf{v}_i^T \mathbf{p} - \varepsilon_i = 0 \quad i = 1, \dots, n-1 \quad (10)$$

This system of equations, as well as the unit norm constraint on the quaternion (cf. Eq. (7)) is quadratic *only* in the elements of \mathbf{q} . However, we can treat it as a *linear* system of equations by introducing new variables for the ten quadratic monomials $q_i q_j$, $i, j = 1, \dots, 4$. Specifically, we stack these monomials in a vector $\bar{\mathbf{q}} = [q_{11} \quad q_{12} \quad \dots \quad q_{44}]^T$, where $q_{ij} := q_i q_j$, and rewrite Eqs. (7) and (10) as the following homogeneous linear system

$$M\bar{\mathbf{q}} + A \begin{bmatrix} {}^1\mathbf{p} \\ {}^2\mathbf{p} \end{bmatrix} - \varepsilon = 0 \Rightarrow \mathcal{M}\bar{\mathbf{x}} = \mathbf{0} \quad (11)$$

where

$$A := \begin{bmatrix} \mathbf{u}_1^T & -\mathbf{v}_1^T \\ \vdots & \vdots \\ \mathbf{u}_{n-1}^T & -\mathbf{v}_{n-1}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \varepsilon := \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{n-1} \\ 1 \end{bmatrix} \quad (12)$$

$$\mathcal{M} := [M \quad A \quad -\varepsilon] \\ \bar{\mathbf{x}} := [\bar{\mathbf{q}}^T \quad {}^1\mathbf{p}^T \quad {}^2\mathbf{p}^T \quad \rho]^T \quad (13)$$

The first $n-1$ rows of M are constructed from the elements of $\mathbf{u}_i^T C \mathbf{v}_i$, after we substitute from Eq. (2), and rearrange the resulting terms. The last row of M , A and ε correspond to the unit quaternion constraint of Eq. (7).

The vector of unknowns, $\bar{\mathbf{x}}_{17 \times 1}$, is a concatenation of the vector $\bar{\mathbf{q}}$, the position vectors ${}^1\mathbf{p}$ and ${}^2\mathbf{p}$, and a scalar, ρ , which we will later constrain to $\rho = 1$. Notice that if $\bar{\mathbf{x}}$ is uniquely specified, this in turn uniquely determines the relative pose ${}^1\mathbf{p}$ and \mathbf{q} . The relative position, ${}^1\mathbf{p}$, can be read directly from $\bar{\mathbf{x}}$, and the elements of \mathbf{q} can be extracted from $\bar{\mathbf{q}}$ using the convention that $q_4 \geq 0$ (which holds without loss of generality, since \mathbf{q} and $-\mathbf{q}$ correspond to the same

rotation [16]). The elements of \mathbf{q} can be extracted from $\bar{\mathbf{q}}$ by setting $q_i = \pm\sqrt{q_{ii}}$, $i = 1, \dots, 4$, with the choice of positive q_4 uniquely determining the sign for the remaining q_i by $\text{sgn}(q_i) = \text{sgn}(q_{i4})$.

The unknown vector $\bar{\mathbf{x}}$ will lie in the nullspace of $\mathcal{M}_{n \times 17}$, or $\bar{\mathbf{x}} \in \ker(\mathcal{M})$. Letting $\boldsymbol{\mu}_i$, $i = 1 \dots N$ denote the N orthonormal basis vectors of $\ker(\mathcal{M})_{17 \times N}$, where $N \geq 17-n$, we can therefore write

$$\bar{\mathbf{x}} = \sum_{i=1}^N \lambda_i \boldsymbol{\mu}_i \quad (14)$$

Notice that a given set of λ_i uniquely defines $\bar{\mathbf{x}}$, and hence the relative pose. Ideally, we would want a one-dimensional nullspace, since then $\bar{\mathbf{x}}$ would follow immediately from the constraint $\rho = 1$. However, using only Eq. (11), this would require at least $n = 16$ distance measurements. With fewer distance measurements, so that the nullspace of \mathcal{M} is of dimension $N > 1$, we need to impose additional constraints on the λ_i 's, as we outline next.

We will restrict ourselves to *homogeneous, quadratic* constraints in the elements x_k , $k = 1 \dots 17$ of $\bar{\mathbf{x}}$. Substituting the appropriate row-elements of Eq. (14), we can write such constraints as

$$\begin{aligned} x_k x_l &= x_{k'} x_{l'} \\ \Leftrightarrow \left(\sum_{i=1}^N \lambda_i \mu_{ik} \right) \left(\sum_{j=1}^N \lambda_j \mu_{jl} \right) &= \left(\sum_{i=1}^N \lambda_i \mu_{ik'} \right) \left(\sum_{j=1}^N \lambda_j \mu_{j'l'} \right) \\ \Leftrightarrow \sum_{i=1}^N \lambda_i^2 (\mu_{ik} \mu_{il} - \mu_{ik'} \mu_{i'l'}) &+ \\ \sum_{i=1}^N \sum_{j=i+1}^N \lambda_i \lambda_j (\mu_{ik} \mu_{jl} + \mu_{il} \mu_{jk} - \mu_{ik'} \mu_{j'l'} - \mu_{i'l'} \mu_{jk'}) &= 0 \end{aligned} \quad (15)$$

where μ_{ik} denotes the k -th element of the vector $\boldsymbol{\mu}_i$.

For the elements of $\bar{\mathbf{q}}$, 20 such quadratic constraints can be established (cf. Appendix), another six constraints are derived from the rows of ${}^1\mathbf{p} = C^2\mathbf{p}$ and ${}^2\mathbf{p} = C^{T1}\mathbf{p}$, and finally one constraint from ${}^1\mathbf{p}^T {}^1\mathbf{p} = d_0^2$, giving a total of 27 *linearly (but not necessarily algebraically) independent* constraints. Notice that the final 7 equations can be written as a homogeneous, quadratic constraint in the elements of $\bar{\mathbf{x}}$ by exploiting the fact that $\rho = \rho^2 = 1$.

Similarly to the linearization of the quadratic equations in the elements of \mathbf{q} , we can express these 27 constraints as a homogeneous system of linear equations in the variables $\bar{\boldsymbol{\lambda}} := [\lambda_{11} \quad \lambda_{12} \quad \dots \quad \lambda_{NN}]^T$, with $\lambda_{ij} := \lambda_i \lambda_j$, as

$$\mathcal{L}\bar{\boldsymbol{\lambda}} = \mathbf{0} \quad (16)$$

The elements of matrix \mathcal{L} are functions only of the known null vectors $\boldsymbol{\mu}_i$, $i = 1 \dots N$. Notice that the dimension of $\bar{\boldsymbol{\lambda}}$ depends on the dimension N of $\ker(\mathcal{M})$, as $\dim(\bar{\boldsymbol{\lambda}}) = \frac{N(N+1)}{2}$. In order for Eq. (16) to have a unique solution, \mathcal{L} must be rank-deficient by exactly one. For this reason, the 27 constraints mentioned above will only be able to uniquely

² $|\cdot|$ is a measure of the size of a set. When the set is continuous, it is the total volume of the set.

specify $\bar{\lambda}$ for $N \leq 7$, in turn requiring $n \geq 10$ distance measurements.

Choosing $\bar{\lambda}$ as the nullvector of \mathcal{L} (or, in the presence of noise, the singular vector corresponding to the smallest singular value), we can recover the λ_i 's up to scale from the elements of $\bar{\lambda}$ by setting $\lambda_i = \pm\sqrt{\lambda_{ii}}$, with the choice of sign for λ_1 determining the sign for the remaining λ_i from the elements of $\bar{\lambda}$, i.e., $\text{sgn}(\lambda_i) = \text{sgn}(\lambda_1)\text{sgn}(\lambda_{1i})$.

We enforce the correct scale of $\bar{\lambda}$ from the constraint

$$\rho = \sum_{i=1}^N \lambda_i \mu_{i\rho} = 1 \quad (17)$$

using the last row of Eq. (14).

Once the coefficients λ_i have been determined, the vector \bar{x} is computed using Eq. (14). Finally, the relative pose, $({}^1\mathbf{p}, \mathbf{q})$, is reconstructed as outlined above.

A necessary condition for this method to work is that the nullspace of \mathcal{M} is of dimension $N \leq 7$, and the nullspace of \mathcal{L} is of dimension 1. Singular configurations (in which one or more degrees of freedom are unobservable from the given measurements) will manifest themselves in the form of matrices \mathcal{M} and \mathcal{L} losing rank. Our algorithm is able to detect such pathological cases, and will require more distance measurements to be taken before a solution is computed. Geometric interpretation or more precise characterization of singular configurations is subject of ongoing work.

VI. WEIGHTED LEAST SQUARES REFINEMENT

In the presence of noise, we are interested in the maximum likelihood estimate of the relative pose. To this end, we refine the relative pose estimate provided by the linear algorithm outlined in Section V, by using it as the initial guess in a Weighted Least-Squares (WLS) algorithm.

We will be using the standard additive error model for the relative position,³

$$\mathbf{p} = \hat{\mathbf{p}} + \tilde{\mathbf{p}} \quad (18)$$

but a multiplicative error model for the quaternion. In particular, if $\hat{\mathbf{q}}$ denotes the estimated quaternion for the relative attitude between the two robots, the true attitude, \mathbf{q} is described by

$$\mathbf{q} = \delta\mathbf{q} \otimes \hat{\mathbf{q}} \quad (19)$$

where \otimes denotes quaternion multiplication. Intuitively, $\delta\mathbf{q}$ describes the small rotation that makes the estimated and the true orientation coincide. Using the small-angle approximation, the error quaternion can be written as

$$\delta\mathbf{q} \simeq \left[\frac{1}{2}\delta\boldsymbol{\theta}^T \quad 1 \right]^T \quad (20)$$

The three-element parameterization $\delta\boldsymbol{\theta}$ is a minimal error representation and thus avoids the loss of rank of the covariance matrix, which would arise in the standard, additive, four-element parameterization due to the unit quaternion constraint.

³We will denote estimated quantities by “ $\hat{\cdot}$ ”, and errors by “ $\tilde{\cdot}$ ”.

The error in the robot positions $\mathbf{u}_i, \mathbf{v}_i$, arising from IMU integration, will be modeled as additive Gaussian, so that

$$\mathbf{u}_i = \hat{\mathbf{u}}_i + \tilde{\mathbf{u}}_i \quad \mathbf{v}_i = \hat{\mathbf{v}}_i + \tilde{\mathbf{v}}_i \quad (21)$$

Notice that in general, the errors for the same robot will be correlated (e.g., $E[\tilde{\mathbf{u}}_i \tilde{\mathbf{u}}_j^T] \neq 0$), but those between robots will be independent ($E[\tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_j^T] = 0$).

Assuming that the distance measurements are corrupted by white, Gaussian noise, we write the measurement equation as

$$z_i = d_i + n_i, \quad (22)$$

with the inter-robot distance, d_i , defined as the 2-norm of vector \mathbf{w}_i (cf. Eq. (4)).

Using the chain rule, we compute the Jacobians as

$$H_{p_i} = \frac{\partial z_i}{\partial \mathbf{1}\mathbf{p}} = \frac{\mathbf{w}_i^T}{d_i} \quad H_{\theta_i} = \frac{\partial z_i}{\partial \boldsymbol{\theta}} = \frac{\mathbf{w}_i^T [C\mathbf{v}_i \times]}{d_i} \quad (23)$$

$$H_{u_i} = \frac{\partial z_i}{\partial \mathbf{u}_i} = -\frac{\mathbf{w}_i^T}{d_i} \quad H_{v_i} = \frac{\partial z_i}{\partial \mathbf{v}_i} = \frac{\mathbf{w}_i^T C}{d_i} \quad (24)$$

and evaluate them at the current linearization point.

The measurement residual can now be approximated as

$$\tilde{z}_i = z_i - \hat{z}_i \quad (25)$$

$$\simeq [H_{\theta_i} \quad H_{p_i}] \begin{bmatrix} \delta\boldsymbol{\theta} \\ \tilde{\mathbf{p}} \end{bmatrix} + H_{u_i} \tilde{\mathbf{u}}_i + H_{v_i} \tilde{\mathbf{v}}_i + n_i \quad (26)$$

Stacking all available distance measurements yields the linearized measurement error

$$\tilde{\mathbf{z}} \simeq H \begin{bmatrix} \delta\boldsymbol{\theta} \\ \tilde{\mathbf{p}} \end{bmatrix} + H_u \tilde{\mathbf{u}} + H_v \tilde{\mathbf{v}} + \mathbf{n} \quad (27)$$

where $H = [H_\theta \quad H_p]$. H_θ and H_p contain the stacked Jacobians H_{θ_i} and H_{p_i} , $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are the stacked robot position error vectors, and the matrices H_u and H_v contain the elements H_{u_i} and H_{v_i} at the appropriate column positions.

Assuming Gaussian measurement noise \mathbf{n} with covariance R , and Gaussian noise in the position estimates $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, with covariances P_u and P_v , we can write the residual covariance (the weighting matrix) as

$$W = H_u P_u H_u^T + H_v P_v H_v^T + R \quad (28)$$

We find the correction by solving the weighted Normal equations [21]

$$H^T W^{-1} H \begin{bmatrix} \delta\boldsymbol{\theta} \\ \tilde{\mathbf{p}} \end{bmatrix} = H^T W^{-1} (\mathbf{z} - \hat{\mathbf{z}}) \quad (29)$$

Note that a prerequisite for computing the correction as well as the covariance matrix is for the measurement matrix H to be of full column rank, thus ensuring local observability. When 10 or more distance measurements are available, global observability is ensured, if matrices \mathcal{M} and \mathcal{L} are of sufficient rank (as described in the previous section).

We update the position as

$${}^1\hat{\mathbf{p}}^{j+1} = {}^1\hat{\mathbf{p}}^j + \tilde{\mathbf{p}} \quad (30)$$

and the quaternion as

$$\delta \mathbf{q} = \left[\frac{1}{2} \delta \boldsymbol{\theta}^T \quad 1 \right]^T, \quad \delta \mathbf{q} = \frac{\delta \mathbf{q}}{\|\delta \mathbf{q}\|} \quad (31)$$

$$\hat{\mathbf{q}}^{j+1} = \delta \mathbf{q} \otimes \hat{\mathbf{q}}^j \quad (32)$$

and iterate to convergence, using the latest estimates, ${}^1\hat{\mathbf{p}}^j$ and $\hat{\mathbf{q}}^j$, to compute H and W .

The final covariance of the estimate is given by

$$E \left[\begin{bmatrix} \delta \boldsymbol{\theta} \\ \delta \hat{\mathbf{p}} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{\theta} \\ \delta \hat{\mathbf{p}} \end{bmatrix}^T \right] = (H^T W^{-1} H)^{-1} \quad (33)$$

Once a pose estimate and its covariance are computed, additional distance measurements can be processed in a recursive estimator, such as an extended Kalman Filter. Notice that, alternatively, the algebraic method as well as the WLS will also work with more than 10 measurements.

Fig. 4 shows an example of the aligned trajectories after the linear algorithm and the subsequent WLS refinement.

VII. SIMULATION RESULTS

We have evaluated the performance of our algorithm in simulations, for different values of odometry and distance measurement noise covariance. In particular, we have generated random trajectories, with the two robots being on average 1 m - 2 m apart, and moving an average of 3 m - 6 m between taking distance measurements. The odometry noise affecting each robot motion between the distance measurements was assumed additive, with covariance $Q = \sigma_p^2 \cdot I$. The measurement noise of the distance measurements was assumed i.i.d., so that $R = \sigma_n^2 \cdot I$. Fig. 3 shows the averaged results of 1000 runs for each noise setting.

Fig. 3(a) shows the error in relative position produced by the algebraic method, as a function of measurement and odometry noise variance. Obviously, in all but the cases of negligible noise, a WLS refinement step is necessary to produce reasonable estimates (note that the algebraic method is exact in the noise-free case). However, any iterative method such as WLS requires an initial estimate, whose quality greatly impacts the speed of convergence and quality of the solution. Poor initial estimates can result in local minima and even divergence. Figs. 3(b), 3(c), and 3(d) compare the position error, the attitude error, and the measurement residual of the algebraic method against those obtained through WLS, once using the algebraic method's result for initialization, and once with a random initial guess. The errors achieved when using the algebraic initialization are consistently smaller than those achieved by the random initial guess.

This finding is further confirmed by the number of iterations required for convergence (using a threshold on the norm of the correction as stopping criterion). Fig. 3(e) shows that using the algebraic result for initialization requires roughly half the number of iterations than that of the random initial guess. Furthermore, a random initial guess leads more often to divergence (cf. Fig. 3(f)). We considered a particular instance of WLS refinement as diverged, if it required more

than 100 iterations. These results underline the advantage of using a closed-form method over iterative numerical solvers. Choosing a large number of random initial estimates to cover the 6 d.o.f configuration space in order to avoid divergence has high computational complexity and is prohibitive in real-time applications.

Using a polynomial solver package (based, for example, on homotopy continuation), such as PHCpack [17], represents another alternative to our method. PHCpack is well suited to solve square systems, i.e., systems consisting of as many equations as unknowns. After parameterizing the attitude using the Rodrigues vector [22] instead of quaternions (reducing the number of unknowns from seven to six), PHCpack was able to solve the system in approximately 25 s on a standard desktop PC, finding all 40 complex and real roots. In order to obtain a unique solution, additional constraints need to be added, making the system non-square (and greatly increasing the run time of PHCpack). Unfortunately, a small amount of noise, even that generated by simple round-off errors, will quickly render a non-square system infeasible. In the presence of measurement noise, one approach to obtain a unique initial guess would be to solve several square subproblems and choose the mean of the closest cluster of real solutions as the initial guess for a WLS refinement (similar to RANSAC). Another approach would be to solve the square system once immediately after the first six distance measurements become available, and use all real results to initialize a multi-hypotheses Kalman Filter. Even though the runtime of PHCpack could probably be improved by methods such as parameter continuation, our algorithm's runtime of 4 ms to obtain the closed form estimate plus on average 50 ms for the WLS, even in a non-optimized Matlab implementation, make our approach highly attractive for real-time robotic applications.

VIII. CONCLUSION AND FUTURE WORK

In this paper we have discussed the problem of computing the 6 d.o.f. relative pose between two moving robots using only robot-to-robot distance measurements. From the mechanical analogue, the general Stewart Platform [2], it is known that six measurements yield 40 discrete, complex solutions. Up to now, there exists no closed-form solution to this problem.

We present, for the first time, an algebraic method using at least 10 distance measurements, which requires *no additional sensors* and *no coordination* of the robot motion, or, analogously, no constraints on the geometry of the Stewart Platform. We further propose an iterative Weighted Least-Squares refinement step to obtain the maximum likelihood estimate in the presence of noise. While our algorithm is able to detect unobservable, singular configurations, their general characterization remains subject of ongoing work.

We believe that our method has great potential, not only in the field of multi-robot navigation, in particular for aerospace and underwater vehicles, but also provides further insight into the kinematics of Stewart Platforms.

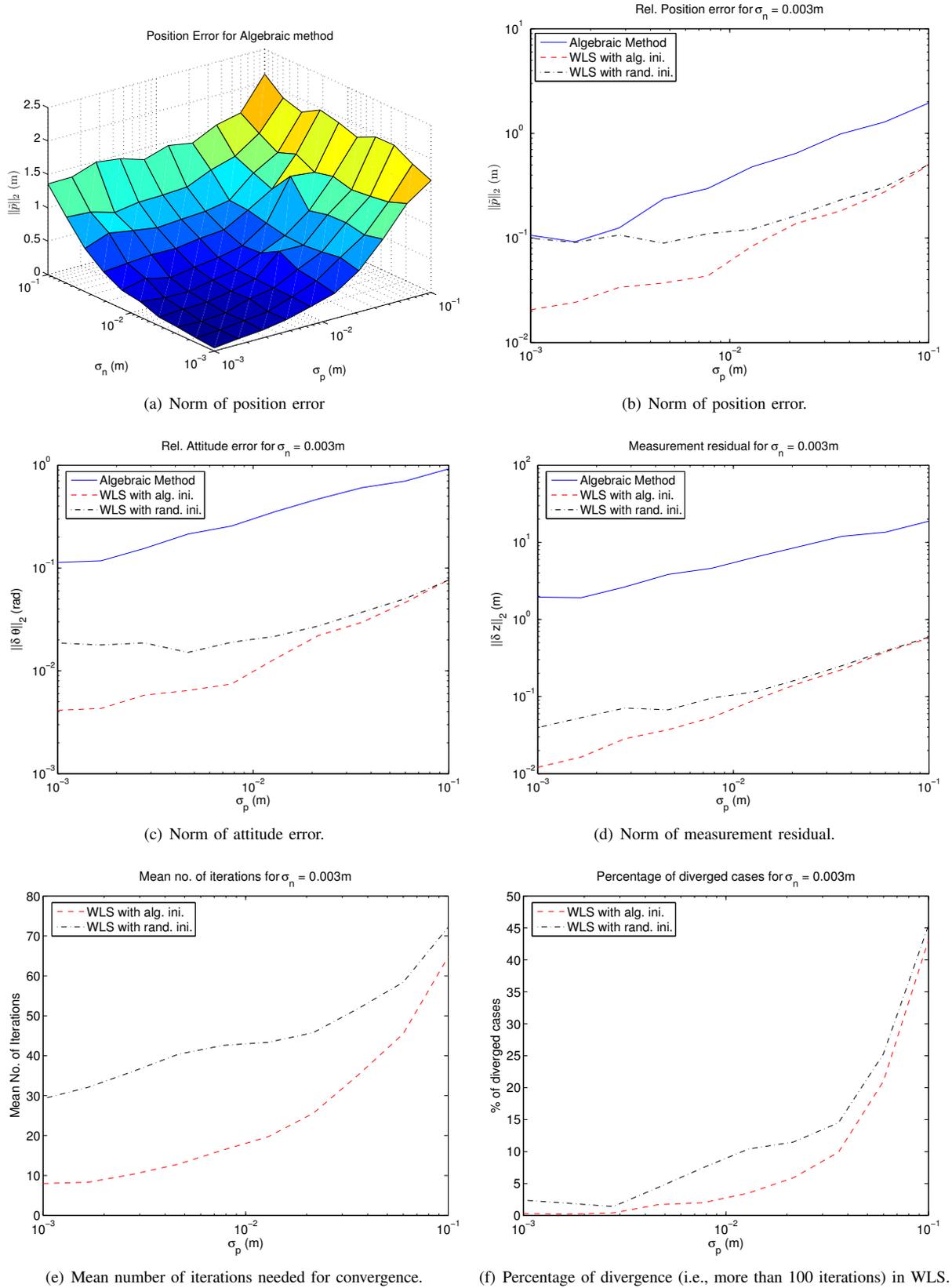


Fig. 3. Simulation results for the algebraic method and WLS refinement (using the result from the algebraic method or a random initialization), as a function of different covariances for measurement and odometry noise. The results for each setting were averaged over 1000 trials. The inter-robot distances were between 1 m-2 m, and the robots moved 3 m-6 m between measurements. Notice that the algebraic method yields competitive results for small errors, and provides good initialization for WLS. Plots 3(b)-3(f) show results for a fixed measurement noise of $3\sigma_n = 1$ cm, as function of odometry noise standard deviation σ_p .

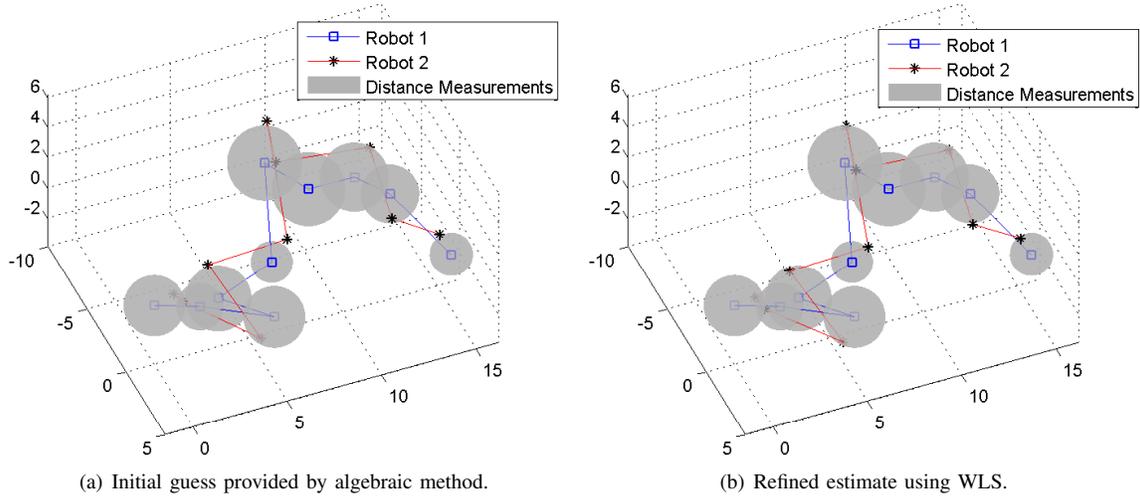


Fig. 4. The two robot trajectories aligned based on the output of the algebraic method (left), and after subsequent refinement using Weighted Least-Squares (right) for a random sample trajectory. The grey spheres denote the range measurements, centered on Robot 1. (Measurement noise $\sigma_n = 0.01$ m, noise in proprioceptive position estimates $\sigma_p = 0.1$ m.)

APPENDIX

Quadratic Constraints in q_i

We use the following notation: Let $q_{ij} = q_i q_j$. It is true that $q_{ij} q_{kl} = q_{i'j'} q_{k'l'}$ for i', j', k', l' a permutation of i, j, k, l . There are 20 such constraints listed below:

$$\begin{aligned}
 q_{11}q_{44} &= q_{14}^2 & q_{14}q_{24} &= q_{12}q_{44} & q_{23}q_{34} &= q_{33}q_{24} \\
 q_{22}q_{44} &= q_{24}^2 & q_{24}q_{34} &= q_{23}q_{44} & q_{13}q_{34} &= q_{33}q_{14} \\
 q_{33}q_{44} &= q_{34}^2 & q_{34}q_{14} &= q_{13}q_{44} & q_{23}q_{24} &= q_{22}q_{34} \\
 q_{11}q_{22} &= q_{12}^2 & q_{12}q_{13} &= q_{11}q_{23} & q_{12}q_{24} &= q_{22}q_{14} \\
 q_{22}q_{33} &= q_{23}^2 & q_{12}q_{23} &= q_{22}q_{13} & q_{13}q_{14} &= q_{11}q_{34} \\
 q_{33}q_{11} &= q_{13}^2 & q_{13}q_{23} &= q_{33}q_{12} & q_{12}q_{14} &= q_{11}q_{24} \\
 & & q_{12}q_{34} &= q_{13}q_{24} & q_{12}q_{34} &= q_{14}q_{23}
 \end{aligned}$$

REFERENCES

- [1] M. S. Grewal, L. R. Weill, and A. P. Andrews, *Global Positioning Systems, Inertial Navigation, and Integration*. John Wiley & Sons, Inc., 2001.
- [2] D. Stewart, "A platform with six degrees of freedom," in *Proc. Inst. Mech. Engrs.*, vol. 180, no. 15, 1965, pp. 371–376.
- [3] L. Doherty, K. S. J. Pister, and L. E. Ghaoui, "Convex position estimation in wireless sensor networks," in *IEEE INFOCOM 2001. Proceedings Twentieth Annual Joint Conference of IEEE Computer and Communications Societies*, Anchorage, AK, April 2001, pp. 1655–1663.
- [4] J. J. Caffery, "A New Approach to the Geometry of TOA Location," in *IEEE Vehicular Technology Conference (VTC)*, vol. 4, Boston, MA, September 24–28 2000, pp. 1943–1949.
- [5] P. Bahl and V. Padmanabhan, "Radar: An in-building RF-based User Location and Tracking System," in *IEEE INFOCOM 2000*, Tel Aviv, Israel, Mar. 2000, pp. 775–784.
- [6] J. Djughash, S. Singh, and P. I. Corke, "Further results with localization and mapping using range from radio," in *International Conference on Field and Service Robotics*, Pt. Douglas, Australia, Jul. 2005, pp. 231–242.
- [7] P. M. Newman and J. J. Leonard, "Pure Range-only Sub-sea SLAM," in *IEEE International Conference on Robotics and Automation*, vol. 2, Taiwan, September 14–19 2003, pp. 1921–1926.
- [8] G. A. Kantor and S. Singh, "Preliminary results in range-only localization and mapping," in *IEEE International Conference on Robotics and Automation*, vol. 2, Washington D.C., May 11–15 2002, pp. 1818–1823.
- [9] S. Higo, T. Yoshimitsu, and I. Nakatani, "Localization on small body surface by radio ranging," in *16th AAS/AIAA Space Flight Mechanics Conference*, Tampa, Florida, January 22–26 2006.
- [10] F. Beutler and U. D. Hanebeck, "Closed-form range-based posture estimation based on decoupling translation and orientation," in *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '05)*, vol. 4, Philadelphia, PA, March 18–23 2005, pp. 989–992.
- [11] C. W. Wampler, "Forward displacement analysis of general six-in-parallel sps (stewart) platform manipulators using soma coordinates," *Mechanism and Machine Theory*, vol. 31, no. 3, pp. 331–337, 1996.
- [12] P. Dietmaier, "The stewart-gough platform of general geometry can have 40 real postures," in *Advances in Robot Kinematics: Analysis and Control*, J. Lenarcic and M. L. Husty, Eds. Kluwer Academic Publishers, 1998, pp. 7–16.
- [13] M. L. Husty, "An algorithm for solving the direct kinematics of general stewart-gough platforms," *Mech. Mach. Theory*, vol. 31, no. 4, pp. 365–380, 1996.
- [14] I. A. Bonev and J. Ryu, "A new method for solving the direct kinematics of general 6-6 stewart platforms using three extra sensors," *Mechanism and Machine Theory*, vol. 35, no. 3, pp. 426–436, 1999.
- [15] X. S. Zhou and S. I. Roumeliotis, "Determining the robot-to-robot relative pose using range-only measurements," in *IEEE International Conference on Robotics and Automation*, Rome, Italy, April 10–14 2007, pp. 4025–4031.
- [16] W. G. Breckenridge, "Quaternions proposed standard conventions," Jet Propulsion Laboratory, Pasadena, CA, Interoffice Memorandum IOM 343-79-1199, 1999.
- [17] J. Verschelde, "Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation," *ACM Transactions on Mathematical Software*, vol. 25, no. 2, pp. 251–276, 1999.
- [18] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, 2nd ed. Springer, 1996.
- [19] A. Ansar and K. Daniilidis, "Linear pose estimation from points or lines," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 25, no. 5, pp. 578–589, May 2003.
- [20] L. Quan and Z.-D. Lan, "Linear n-point camera pose determination," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 21, no. 8, pp. 774–780, 1999.
- [21] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Upper Saddle River, New Jersey: Prentice Hall, 1993.
- [22] M. D. Shuster, "A survey of attitude representations," *Journal of the Astronautical Sciences*, vol. 41, no. 4, pp. 439–517, October–December 1993.