Optimal Formations for Cooperative Localization of Mobile Robots

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***Abstract*— This paper studies the effects of the geometry of a mobile robot formation on the accuracy of the robots' localization. The general case of heterogeneous (in terms of sensor accuracy) robot teams performing Cooperative Localization is considered. An analysis of the time evolution of the covariance matrix of the position estimates allows us to express the steady-state positioning uncertainty of the robots as an *analytic function* of the relative positions of the robots in the formation. This metric encapsulates the effect of formation geometry on the information content of the exteroceptive measurements, as well as the effect of the influx of uncertainty due to the errors in the robots' odometry. Thus, by minimizing the trace of the steady state covariance matrix with respect to the positions of the robots, the optimal robot configuration can be determined. Numerical experiments are presented, which indicate that it is possible to derive a practical rule for determining optimal formations, without the need to resort to extensive simulations, or experimentation.**

I. Introduction

The topic of *Cooperative Localization* in multi-robot systems has recently attracted the interest of many researchers due to the greater versatility that robotic teams provide. Numerous algorithms for efficient pose estimation in robot teams have been proposed in the literature, such as Extended Kalman Filtering [1], Particle Filtering [2], Maximum Likelihood estimation [3], and Set Membership approaches [4]. In this paper, we specifically address the issue of localization in mobile robot formations. Our motivation arises from the fact that a large number of applications require robots to move in a coordinated fashion, in order to accomplish a certain task.

The key contributions of this paper are (i) formulating a cost metric and (ii) determining a simple empirical rule for optimal mobile robot formations. Our cost metric employs the trace of the robot state covariance, where the optimality is defined as minimizing the sum of the robots' position variances. This metric is more robust compared to the one given by the determinant of the robot state covariance used in [5] and [6]. For example, in [7] and within the context of Simultaneous Localization and Mapping (SLAM), the determinant of the state covariance converges to zero in the limit when at least one of the landmarks is observed numerous times. Similarly, when robots perform cooperative localization and the estimates of at least two of them become fully correlated, the determinant of the multirobot system covariance matrix will be close to zero.

Our metric computes the cost of robot formations in motion. That is, the robot formation is evaluated at the steady state of the Riccati recursion. The Relative Position Measurement Graph (RPMG) is used to interpret the robot-to-robot relative position detectability. RPMG is a graph whose vertices represent robots in the group and its directed edges correspond to relative position measurements. These are important since we can assess the formation in advance of the mobile robot operation. The results of this work provide important insights about the optimal robot formations. It is shown that the robots must be located at equilateral positions, circumscribed by circles, that define the forbidden vicinity due to the measurement model and the robot physical size.

II. Related Work

Current research on mobile robot formations mainly focuses on the issue of motion control for formation stabilization. The proposed approaches include behavior-based control techniques [8], [9], methods relying on potential fields [10], the use of virtual structures to describe the formation [11], and approaches employing notions of graph theory for characterizing the interconnections between the robots of a team [12], [13]. Additionally, formation design for accomplishing a certain task, such as object moving, has attracted the interest of several researchers [14], [15]. However, the effects of the geometry of a formation on the localization accuracy of its members has, in general, been overlooked, with only a few exceptions.

Specifically, in [16] a robot team comprised of one master and two slave robots is studied and a *portable landmarks-based* technique is adopted, i.e., at each time instant at least one robot remains stationary. The robots move along a straight-line path and record measurements of their relative positions at evenly spaced intermediate points. The authors propose a method for determining the optimal relative positions between the robots that attain the maximum possible localization accuracy at the end of the path and identify three configurations that yield superior results. In [6], the robots are allowed to move continuously towards a target configuration and a gradient-based optimization method is employed for determining the trajectories that yield optimal localization performance. The presented numerical experiments indicate that improved positioning accuracy is achieved when the robots of the team do *not* follow identical trajectories. We note, however, that the aforementioned techniques are not applicable in cases where motion in formation is required, since the relative positions of the robots vary substantially as the team moves.

The impact of the geometry of a *static* robot formation on the accuracy of pose estimation is studied in the work of Zhang et al. [17]. The authors consider formations of robots that receive both absolute position measurements and relative measurements (i.e., relative range, bearing, or orientation) and derive a necessary condition on the number of measurements of each type, in order for the formation to be localizable. A study of the structure of the measurement matrix shows that the information matrix corresponding to the exteroceptive measurements is a function of the relative positions of the robots. Consequently, it is meaningful to seek optimal configurations for the formation. A gradient-based optimization technique is employed to determine local maxima of the trace of the information matrix. However, due to the non-concavity of the objective function, the selected optimization method does not guarantee global optimality of the solution.

Our work differs from the aforementioned approaches, in that we consider a team of robots that *move* while maintaining their formation. The positioning accuracy of the robots is expressed as an *analytic function* of their relative positions and this enables us to determine, by employing numerical optimization, the optimal geometry for a robot team with a given (and possibly heterogeneous) set of sensors.

III. Problem Formulation

Consider a group of N robots that move in formation towards a goal position. The robots employ an Extended Kalman Filter (EKF) estimator to perform CL, using proprioceptive (i.e., velocity) measurements to propagate their position estimates, and exteroceptive measurements (i.e., robot-to-robot relative position measurements and absolute position measurements) to update these estimates. Additionally, we assume that each robot receives measurements of its absolute orientation, such as those provided by a compass or a sun sensor, or those inferred by exploiting the perpendicularity of the walls in an indoor environment. Availability of such measurements is required, in order to maintain the variance of the orientation estimates of the robots within finite limits. Without the absolute orientation update, the errors in the robots' orientation estimates will grow unbounded and any EKF-based estimator of their position will eventually diverge [18].

Since the robots in the team have access to absolute position information, the covariance matrix of their position estimates converges to a constant steady-state value, after an initial transient phase [19]. Our goal is to determine the optimal geometry of the formation, in order to minimize the trace of the steady state covariance matrix. This is facilitated by decoupling the task of orientation estimation from that of position estimation. Specifically, a state vector comprised of the positions of the N robots is defined, and the velocity and orientation of each robot are treated as measurement inputs for propagating the state estimates of the EKF. At each time step, a set of exteroceptive (relative and absolute position) measurements becomes available and is processed in order to update the position estimates of the robots. This formulation enables us to express the steady state covariance matrix of the robots' position estimates as an *analytic function* of the positions of the robots and results in an analytical expression for the objective function.

The basis of our approach for determining the steady state value of the robots' positioning uncertainty is the use of the Riccati recursion (cf. Eq. (19)), that describes the time evolution of the estimates' covariance in the EKF framework. In the case of a robot team moving in formation, the matrix coefficients in the Riccati recursion are approximately time-invariant. This allows us to compute the steady state solution of the recursion, by solving a constant coefficient algebraic Riccati equation (cf. Eq. (20)). In the following sections the motion and measurement models for the robot team are described, and the cost function to be minimized is derived as a function of the robots' relative positions in the formation.

A. Position propagation

The discrete-time kinematic equations for a mobile robot moving in 2D are

$$x_i(k+1) = x_i(k) + V_i(k)\delta t\cos(\phi_i(k)) \quad (1)$$
$$y_i(k+1) = y_i(k) + V_i(k)\delta t\sin(\phi_i(k)) \quad (2)$$

where $V_i(k)$ denotes the translational velocity of the i-th robot at time step k, and δt is the sampling period. In the Kalman filter framework, the estimates of the robot's position are propagated using the measurements of the robot's velocity, $V_{m_i}(k)$, and the estimates of the robot's orientation, $\hat{\phi}_i(k)$. Clearly, these equations are time varying and nonlinear due to the dependence on the robot's orientation estimates. By linearizing Eqs. (1) and (2) the error propagation equation for the robot's position is readily derived:

$$\begin{bmatrix} \widetilde{x}_{i_{k+1|k}} \\ \widetilde{y}_{i_{k+1|k}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_{i_{k|k}} \\ \widetilde{y}_{i_{k|k}} \end{bmatrix} + \begin{bmatrix} \delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k)) \\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k)) \end{bmatrix} \begin{bmatrix} w_{V_i}(k) \\ \widetilde{\phi}_i(k) \end{bmatrix}$$
$$\Leftrightarrow \quad \widetilde{X}_{i_{k+1|k}} = I_{2\times 2}\widetilde{X}_{i_{k|k}} + G_i(k)W_i(k) \quad (3)$$

where[1] $w_{V_i}(k)$ is a zero-mean white Gaussian noise sequence with variance $\sigma^2_{V_i}$, affecting the velocity measurements, and $\widetilde{\phi}_i(k)$ is the error in the robot's orientation estimate at time k. This is modelled as a zero-mean white Gaussian noise sequence, whose variance, $\sigma^2_{\phi_i}$, is determined by the accuracy of the sensor used for measuring orientation.

From Eq. (3), we deduce that the covariance matrix of the system noise affecting the i-th robot is

$$\begin{aligned} Q_i(k) &= E\{G_i(k)W_i(k)W_i^T(k)G_i^T(k)\} \\ &= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2\sigma^2_{V_i} & 0 \\ 0 & \delta t^2 V^2_{m_i}(k)\sigma^2_{\phi_i} \end{bmatrix} C^T(\hat{\phi}_i(k)) \end{aligned}$$

where $C(\hat{\phi}_i)$ denotes the rotation matrix associated with $\hat{\phi}_i$.

At this point, we note that since the robots move in a predefined formation, all robots are required to move in the same direction and with the same velocity, both of which are considered known constants. Assuming that a motion

[1] Throughout this paper, $\mathbf{0}_{m\times n}$ denotes the $m \times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ the $m \times n$ matrix of ones, and $I_{n\times n}$ the $n \times n$ identity matrix.

controller is used in order to minimize the deviations from the desired formation, and that the accuracy of the velocity measurements and orientation estimates is sufficiently high, we can replace the quantities $V_{m_i}(k)$ and $\hat{\phi}_i(k)$ in the above expression by their respective desired values, V_o and ϕ_o, and thus

$$Q_i(k) \simeq C(\phi_o) \begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_o^2 \sigma_{\phi_i}^2 \end{bmatrix} C^T(\phi_o) = Q_{i_o} \tag{4}$$

The state vector $X(k)$ for the entire system is defined as the stacked vector comprising of the positions of the N robots. The state transition matrix for the entire system at time step k is $\mathbf{\Phi}_k = I_{2N\times 2N}$ and the covariance matrix of the system noise is approximately equal to

$$\mathbf{Q}_o = \begin{bmatrix} Q_{1_o} & \cdots & \mathbf{0}_{2\times 2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & Q_{N_o} \end{bmatrix} \tag{5}$$

which is a constant matrix. The equation for propagating the covariance matrix of the state error is now written as

$$\mathbf{P}_{k+1|k} = \mathbf{P}_{k|k} + \mathbf{Q}_o \tag{6}$$

B. Exteroceptive Measurement Model

In this section we derive the covariance update equation of the EKF. We consider the robot-to-robot as well as the absolute position measurements recorded by the robots and show that the information contributed by the exteroceptive measurements, at each update step, can be expressed as a function of the relative positions of the robots in the formation.

The relative position measurement z_{ij} between robots i and j is defined as

$$z_{ij} = C^T(\phi_i)\,(X_j - X_i) + n_{z_{ij}} = C^T(\phi_i)\Delta p_{ij} + n_{z_{ij}} \tag{7}$$

where $n_{z_{ij}}(k)$ is a white zero-mean Gaussian noise process affecting the measurement. By linearizing this expression, the measurement error equation is obtained:

$$\widetilde{z}_{ij}(k+1) = H_{ij}(k+1)\widetilde{X}_{k+1|k} + \Gamma_{ij}(k+1) n_{ij}(k+1)$$

where

$$H_{ij}(k+1) = C^T(\hat{\phi}_i(k+1)) H_{o_{ij}} \tag{8}$$

$$H_{o_{ij}} = \begin{bmatrix} 0_{2\times 2} & .. & \underbrace{-I_{2\times 2}}_{i} & .. & \underbrace{I_{2\times 2}}_{j} & .. & 0_{2\times 2} \end{bmatrix}$$

$$\widetilde{X}_{k+1|k} = \begin{bmatrix} \widetilde{X}_1^T & .. & \widetilde{X}_i^T & .. & \widetilde{X}_j^T & .. & \widetilde{X}_N^T \end{bmatrix}^T_{k+1|k}$$

$$\Gamma_{ij}(k+1) = \begin{bmatrix} I_{2\times 2} & -C^T(\hat{\phi}_i(k+1)) J \widehat{\Delta p}_{ij}(k+1) \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad n_{ij}(k+1) = \begin{bmatrix} n_{z_{ij}}(k+1) \\ \widetilde{\phi}_i(k+1) \end{bmatrix}$$

$$\widehat{\Delta p}_{ij}(k+1) = \widehat{X}_{j_{k+1|k}} - \widehat{X}_{i_{k+1|k}}$$

The covariance of the error in this measurement is given by

$$\begin{aligned} {}^iR_{jj}(k+1) &= \Gamma_{ij}(k+1) E\{n_{ij}(k+1) n_{ij}^T(k+1)\} \Gamma_{ij}^T(k+1) \\ &= R_{z_{ij}}(k+1) + R_{\tilde{\phi}_{ij}}(k+1) \end{aligned} \tag{9}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_{ij}(k+1)$. More specifically, $R_{z_{ij}}(k+1)$ is the covariance component attributed to the measurement noise $n_{z_{ij}}(k+1)$, and $R_{\tilde{\phi}_{ij}}(k+1)$ is the additional covariance term due to the error $\tilde{\phi}_i(k+1)$ in the orientation estimate of the measuring robot.

Assuming that each relative position measurement is comprised of a distance measurement ρ_{ij} and a bearing measurement θ_{ij}, affected by two independent zero-mean white Gaussian noise sequences n_{ρ_i} and n_{θ_i} respectively, the term $R_{z_{ij}}(k+1)$ can be expressed as [20]:

$$R_{z_{ij}} = C^T(\hat{\phi}_i) \left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2} \widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T + \sigma_{\theta_i}^2 J \widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T J^T \right) C(\hat{\phi}_i)$$

where time indices have been dropped for simplicity, and the variances of the noise in the distance and bearing measurements are denoted as $\sigma_{\rho_i}^2 = E\{n_{\rho_i}^2\}$ and $\sigma_{\theta_i}^2 = E\{n_{\theta_i}^2\}$ respectively.

The error in the orientation estimate of the measuring robot introduces an additional error component to all relative position measurements recorded by the robot and renders them correlated. It can be shown that the additional covariance term for each measurement is equal to [20]:

$$R_{\tilde{\phi}_{ij}}(k+1) = \sigma_{\phi_i}^2 C^T(\hat{\phi}_i) J \widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T J^T C(\hat{\phi}_i) \tag{10}$$

while the matrix of correlation between the errors in the measurements $z_{ij}(k+1)$ and $z_{i\ell}(k+1)$ is

$$\begin{aligned} {}^iR_{j\ell}(k+1) &= \Gamma_{ij}(k) E\{n_{ij}(k+1) n_{i\ell}^T(k+1)\} \Gamma_{i\ell}^T(k) \\ &= \sigma_{\phi_i}^2 C^T(\hat{\phi}_i) J \widehat{\Delta p}_{ij} \widehat{\Delta p}_{i\ell}^T J^T C(\hat{\phi}_i) \end{aligned} \tag{11}$$

The results of Eqs. (9)-(11) allow for the evaluation of the $2M_i \times 2M_i$ covariance matrix $\boldsymbol{R}_i(k+1)$ of all the M_i relative position measurements gathered by robot i at each time instant. This is a matrix whose 2×2 block diagonal elements equal ${}^iR_{ij}(k+1),\ j \in \mathcal{N}_{M_i} \subset \{1,\ldots,N\}$, where $\mathcal{N}_{M_i}$ is the set of the indices of the robots j observed by robot i. The off-diagonal block elements of $\boldsymbol{R}_i(k+1)$ are ${}^iR_{j\ell}(k+1),\ j,\ell \in \mathcal{N}_{M_i}, j \neq \ell$. Simple algebraic manipulation yields the expression

$$\boldsymbol{R}_i(k+1) = \mathbf{D}_{\hat{\phi}_i}^T \boldsymbol{R}_{o_i}(k+1) \mathbf{D}_{\hat{\phi}_i} \tag{12}$$

where $\mathbf{D}_{\hat{\phi}_i} = I_{M_i\times M_i} \otimes C(\hat{\phi}_i)$, with $\otimes$ denoting the Kronecker matrix product, and

$$\begin{aligned} \boldsymbol{R}_{o_i}(k+1) &= \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} - B_i \,\mathrm{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right) B_i^T \\ &+ \sigma_{\theta_i}^2 B_i B_i^T + \sigma_{\phi_i}^2 B_i \mathbf{1}_{M_i\times M_i} B_i^T \end{aligned} \tag{13}$$

In the last expression $B_i = \mathbf{Diag}\left(J\widehat{\Delta p}_{ij_{k+1|k}}\right)$ is a $2M_i \times M_i$ block diagonal matrix with elements $J\widehat{\Delta p}_{ij_{k+1|k}}$, $j \in \mathcal{N}_{M_i}$.

The measurement matrix describing the relative position measurements performed by robot i at each time step is a matrix whose block rows are H_{ij}, and

$$\boldsymbol{H}_i(k+1) = \mathbf{D}_{\hat{\phi}_i}^T \boldsymbol{H}_{o_i} \tag{14}$$

where $\boldsymbol{H}_{o_i}$ is a *constant* matrix with block rows $H_{o_{ij}}$, $j \in \mathcal{N}_{M_i}$ (cf. Eq. (8)). Hence, the information contributed by the measurements of robot i is

$$\boldsymbol{H}_i^T(k+1)\boldsymbol{R}_i^{-1}(k+1)\boldsymbol{H}_i(k+1) = \boldsymbol{H}_{o_i}^T\boldsymbol{R}_{o_i}^{-1}(k+1)\boldsymbol{H}_{o_i}$$

For a known topology of the RPMG and given the accuracy of the robot's exteroceptive sensors, this matrix depends only on the estimates for the relative positions of the robots. However, since the configuration of the robots in the formation remains approximately constant (and assuming small estimation errors), we can approximate the matrix $\boldsymbol{R}_{o_i}(k+1)$ by a constant matrix, obtained by replacing the estimates for the robots' relative positions by their predefined values, i.e.,

$$\begin{aligned} \boldsymbol{R}_{o_i} &\simeq \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} - B_{o_i}\,\mathrm{diag}\left(\frac{\sigma_{\rho_i}^2}{\rho_{o_{ij}}^2}\right)B_{o_i}^T \\ &+ \sigma_{\theta_i}^2 B_{o_i}B_{o_i}^T + \sigma_{\phi_i}^2 B_{o_i}\mathbf{1}_{M_i\times M_i}B_{o_i}^T \end{aligned} \tag{15}$$

where the matrix $B_{o_i} = \mathbf{Diag}\left(J\Delta p_{o_{ij}}\right)$, depends on the values for the relative positions $\Delta p_{o_{ij}}$ between the robots, as they are determined by the geometry of the formation.

In addition to the relative position measurements recorded by the robots, we assume that at least one of the members of the team has access to absolute positioning information. If robot ℓ receives absolute position measurements, the corresponding block row of the measurement matrix is:

$$\boldsymbol{H}_{\alpha_\ell} = \begin{bmatrix} 0_{2\times 2} & \cdots & \underbrace{I_{2\times 2}}_{\ell} & \cdots & 0_{2\times 2} \end{bmatrix} \tag{16}$$

while $\boldsymbol{R}_{\alpha_\ell}$, the covariance of the absolute position measurement, is a constant provided by the specifications of the absolute positioning sensor. In the following, we denote by $\mathcal{N}_\alpha$ the set of indices corresponding to robots receiving absolute position measurements.

The measurement matrix for the entire system, $\mathbf{H}(k+1)$, is defined as a matrix with block rows $\boldsymbol{H}_i(k+1)$, $i = 1..N$ and $\boldsymbol{H}_{\alpha_\ell}$, $\ell \in \mathcal{N}_\alpha$. Since the measurements recorded by different robots are independent, the covariance matrix $\mathbf{R}(k+1)$ of all the exteroceptive measurements is a block diagonal matrix with elements $\boldsymbol{R}_i$, $i = 1..N$, and $\boldsymbol{R}_{\alpha_\ell}$, $\ell \in \mathcal{N}_\alpha$, respectively. Thus the total information available to the estimator at each update step is

$$\begin{aligned} &\mathbf{H}^T(k+1)\mathbf{R}^{-1}(k+1)\mathbf{H}(k+1) \\ &= \sum_i \boldsymbol{H}_i^T(k+1)\boldsymbol{R}_i^{-1}(k+1)\boldsymbol{H}_i(k+1) + \sum_\ell \boldsymbol{H}_{\alpha_\ell}^T\boldsymbol{R}_{\alpha_\ell}^{-1}\boldsymbol{H}_{\alpha_\ell} \\ &\simeq \sum_i \boldsymbol{H}_{o_i}^T\boldsymbol{R}_{o_i}^{-1}\boldsymbol{H}_{o_i} + \sum_\ell \boldsymbol{H}_{\alpha_\ell}^T\boldsymbol{R}_{\alpha_\ell}^{-1}\boldsymbol{H}_{\alpha_\ell} \\ &= \mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o \end{aligned} \tag{17}$$

where $\mathbf{H}_o$ is a matrix with block rows $\boldsymbol{H}_{o_i}$, $i = 1..N$ and $\boldsymbol{H}_{\alpha_\ell}$, $\ell \in \mathcal{N}_\alpha$, while $\mathbf{R}_o$ is a block diagonal matrix with submatrix elements $\boldsymbol{R}_{o_i}$, $i = 1..N$ and $\boldsymbol{R}_{\alpha_\ell}$, $\ell \in \mathcal{N}_\alpha$. Hence, the covariance update equation of the EKF is written as

$$\begin{aligned} \mathbf{P}_{k+1|k+1} &= \left(\mathbf{P}_{k+1|k}^{-1} + \mathbf{H}^T(k+1)\mathbf{R}^{-1}(k+1)\mathbf{H}(k+1)\right)^{-1} \\ &\simeq \left(\mathbf{P}_{k+1|k}^{-1} + \mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o\right)^{-1} \end{aligned} \tag{18}$$

IV. Optimality Criterion

The time evolution of the covariance matrix of the position estimates of the robots is described by the Riccati recursion, which is derived by combining the covariance propagation and update equations (Eqs. (6) and (18)) into a single recursion. The notation is simplified by setting $\mathbf{P}_{k+1|k} = \mathbf{P}_k$, and $\mathbf{P}_{k+2|k+1} = \mathbf{P}_{k+1}$, which yields the recursion

$$\mathbf{P}_{k+1} = \left(\mathbf{P}_k^{-1} + \mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o\right)^{-1} + \mathbf{Q}_o \tag{19}$$

It is important to note that the matrix coefficients $\mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o$ and $\mathbf{Q}_o$ that appear in this recursion are *constant*. Additionally, since the robot team has access to absolute positioning information, the system under consideration is observable [1]. Therefore the Riccati recursion in the last expression converges to a *constant* value at steady state. This value is determined by setting $\mathbf{P}_{k+1} = \mathbf{P}_k = \mathbf{P}_\infty$, and solving the equation

$$\mathbf{P}_\infty = \left(\mathbf{P}_\infty^{-1} + \mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o\right)^{-1} + \mathbf{Q}_o \tag{20}$$

The intermediate steps of the solution involve only algebraic manipulation, and cannot be included in this paper due to space limitations. The resulting solution is

$$\mathbf{P}_\infty = \mathbf{Q}_o^{1/2}\mathbf{U}\,\mathrm{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}\right)\mathbf{U}^T\mathbf{Q}_o^{1/2} \tag{21}$$

where $\mathbf{Q}_o^{1/2}$ denotes the square root matrix of $\mathbf{Q}_o$, and the quantities $\mathbf{U}$ and λ_i, $i = 1,..,2N$ are defined as the matrix of eigenvectors and the eigenvalues, respectively, of matrix

$$\mathbf{C} = \mathbf{Q}_o^{1/2}\mathbf{H}_o^T\mathbf{R}_o^{-1}\mathbf{H}_o\mathbf{Q}_o^{1/2}$$

We observe that the steady state localization accuracy of the robot team is obtained in *analytic form*, and depends on the configuration of the formation (affecting matrices $\mathbf{R}_o$ and $\mathbf{Q}_o$), as well as on the accuracy of the robots' sensors and the topology of the RPMG. Thus, for a robot team whose members receive a specified set of exteroceptive measurements at each time step, the optimal placement of the robots in space, in the sense of attaining the maximum possible positioning accuracy, can be determined by numerical optimization of an appropriate objective function.

Clearly, in order to improve the positioning accuracy of the robot team, the steady state covariance of the robots' position estimates should be minimized. However, $\mathbf{P}_\infty$ is a $2N \times 2N$ matrix and in order to employ numerical optimization a scalar objective function is required. Several scalar performance metrics can be defined based on $\mathbf{P}_\infty$ (e.g., its determinant, its maximum eigenvalue, its trace). In this work, we have selected the trace of $\mathbf{P}_\infty$ as the cost function to be minimized, i.e., we solve the following minimization problem:

$$\text{minimize } \operatorname{trace}(\mathbf{P}_\infty) \tag{22}$$

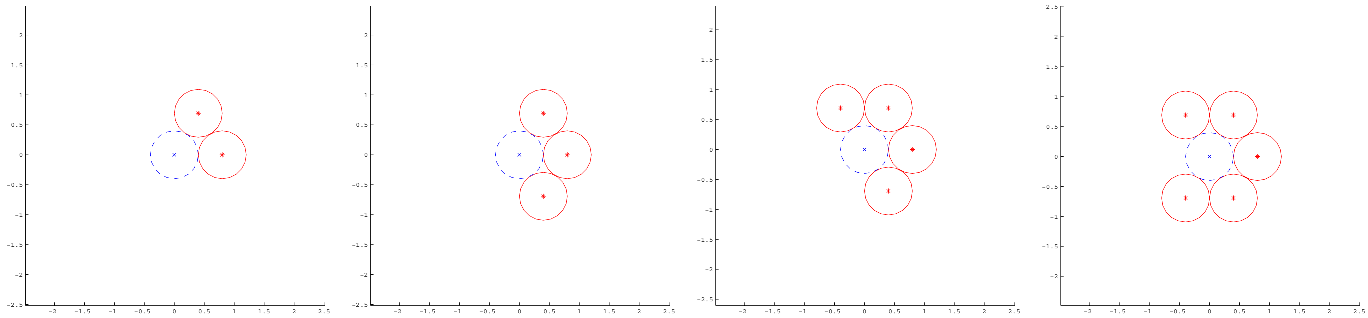

Fig. 1. Optimization results for teams comprising of 3, 4, 5, and 6 robots. The direction of motion for the formation is along the x axis. Robot positions are denoted by crosses (asterisks), which correspond to robots with (without) absolute position sensor measurements. Circles of radii $0.5d_{\min}$ have been superimposed on the figures, to emphasize the fact that the robots lay on the vertices of equilateral triangles.

One reason for selecting the trace of the covariance matrix as the performance metric is that, by doing so, we assign equal importance to the positioning accuracy of all robots. Moreover, we note that the criterion of optimality in the derivation of the Kalman filter estimator is the minimization of the mean squared error of the state estimate which is equivalent to minimizing the trace of the covariance matrix [21].

V. Optimization Results

A Genetic Algorithm (GA) for numerical optimization was employed for the minimization of the objective function in Eq. (22). This selection was stipulated by the fact that the cost metric is a highly nonlinear, non-convex function of the design variables (i.e., the positions of the robots). This renders any gradient-optimization method inappropriate for this problem, since such a method cannot provide any guarantee of global optimality of the solution. Moreover, in order to take into consideration the physical dimensions of the robots, we need to impose constraints on the minimum allowable distance between any two robots. These constraints are nonlinear and their implementation is not trivial in a gradient-based method. Evolutionary algorithms do not suffer from the existence of local minima in the cost function, and allow for easy incorporation of the nonlinear constraints in the optimization. This is achieved by modifying the cost function, in a way that yields a (practically) infinite cost for the solutions that violate the constraints. Thus, GAs are a suitable tool for the problem at hand. In our implementation, the Genetic Algorithm Optimization Toolbox (GAOT), which is open to the public under GNU General Public License (GPL) has been used. The technical description of the GAOT is found in [22].

In order to demonstrate the generality of our approach, the optimization results of this section pertain to heterogeneous robot teams. Specifically, the standard deviation of the orientation estimates of the robots varies between $\sigma_{\phi_{\min}} = 1^{o}$ and $\sigma_{\phi_{\max}} = 4^{o}$, while the standard deviation of the velocity measurements varies between $\sigma_{V_{\min}} = 0.01$m/sec. and, $\sigma_{V_{\max}} = 0.1$m/sec. The formation is moving at a constant speed of $V_o = 1$m/sec, and for simplicity, a complete RPMG topology is employed for the exteroceptive measurements, i.e., each robot measures the relative position of all other robots in the team. Each relative position measurement comprises of a range and a bearing measurement, whose standard deviations are given by $(\sigma_\rho, \sigma_\theta) = (0.05\text{m}, 3^{o})$ for all robots. One of the robots is equipped with an absolute position sensor, that provides measurements whose errors across each axis are assumed to be uncorrelated, zero-mean, white Gaussian noise sequences with standard deviation $\sigma_\alpha = 0.20$m. Note that only *one* absolute position sensor for the team is sufficient in order to assure that *all* robots have bounded positioning uncertainty.

In Fig. 1, the optimal geometries of formations comprising of 3, 4, 5, and 6 robots are shown. In order to account for the physical dimensions of the robots, as well as for the fact that sensors based on time-of-flight measurements have a minimum allowable sensing range, we impose a minimum distance constraint of $d_{\min} = 0.8$m between any two robots. From these figures it becomes apparent that the optimal configuration arises when the robots remain as close together as possible. The resulting geometry is such that the robots lie on the vertices of adjacent equilateral triangles, whose sides are equal to $d_{\min}$. This observation is general and similar results have been obtained in all our tests with teams of increasing size and varying accuracy of their sensors. These results make sense intuitively, since the information content of the bearing measurements decreases as the distance between robots increases. Therefore, it is beneficial for the robots to remain as close as possible. An additional observation is that the robot which is equipped with absolute position sensing capabilities is positioned on the vertex of the formation that has the minimum sum of distances to the rest of the vertices. This is due to the fact that the robot which has access to absolute position measurements can estimate its position with greater accuracy. Therefore the relative position measurements that involve this robot convey more significant information.

At this point, a remark about the nature of the cost function is due. In order to derive a closed form expression for the objective function, we have approximated the time varying estimates of the position and orientation of the robots with constant values, determined by the geometry of the formation. To verify that this approximation is valid, in Fig. 2 the time evolution of $\mathrm{trace}(\mathbf{P}_k)$ is plotted and compared to its theoretically computed steady state value, $\mathrm{trace}(\mathbf{P}_\infty)$, for a robot team comprising of 5 robots. We observe that after an initial transient phase, the trace of

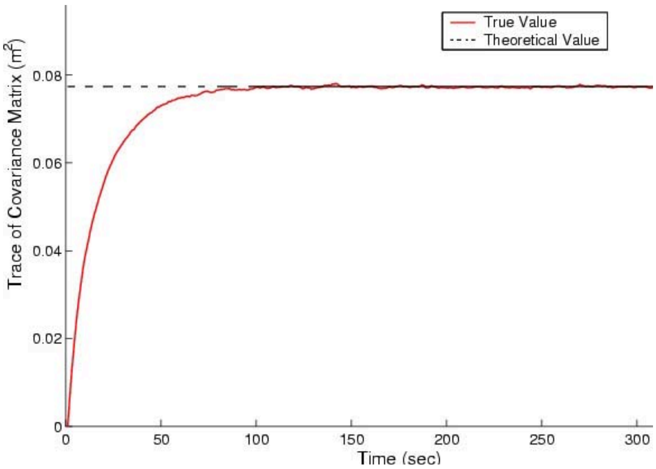

Fig. 2. Comparison of the actual trace of the covariance matrix, during EKF-based localization, with the theoretically computed value, which is based on the desired robot positions.

the covariance fluctuates around a value very close to the theoretical one. The deviations, attributed to the errors in the estimates of the position and orientation of the robots, are not significant, and thus we conclude that the objective function of Eq. (22) yields an accurate description of the formation positioning performance.

VI. Conclusions

In this paper we propose the first, to the best of our knowledge, method for optimizing the geometry of *mobile* robot formations in order to attain maximum localization accuracy. We consider the general case of heterogeneous (in terms of sensor accuracy) robot teams, whose members perform Cooperative Localization, with an arbitrary RPMG topology. The performance metric we propose is the trace of the steady-state covariance of the robots' position estimates, which can be evaluated as an *analytic function* of the relative positions of the robots in the formation. This metric encapsulates the effect of formation geometry on the information content of the exteroceptive measurements, as well as, the effect of the influx of uncertainty due to the errors in the robots' odometry. Numerical optimization of the resulting objective function indicates that in the case that all robots are able to measure the relative positions of all other robots, the optimal configuration requires all robots to be placed as close as possible. Since practical limitations prevent the robots from approaching each other below some given minimum distance, the resulting formation geometry comprises of adjacent equilateral triangles. This is an important observation, that can be employed as a practical rule for determining optimal formations, without the need to resort to extensive simulations, or experimentation.

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