A is SPD
\[ \hat{x} = x + \alpha r \]
\( \alpha \) determined by the condition:

\[ \hat{r} = b - (Ax + \alpha r) = r - \alpha A r \perp r \implies (r, r) - \alpha (Ar, r) = 0 \implies \alpha = (r, r)/(Ar, r) \]

we are trying to minimize
\[ \frac{1}{2} (Ax, x) - (b, x) = \| x - \cdot \|_A^2 + \text{constant} \]
\[ \frac{1}{2} (Ax, x) - (b, x) = \text{constant} = \text{level curve} \]

\[ \hat{r} = \hat{r} = b - (Ax + \alpha r) = r - \alpha A r = (I - \alpha A) r \]
\[ \hat{p}_{k+1} = (1 - \alpha_k t)(1 - \alpha_{k-1} t) \ldots (1 - \alpha_0 t) \]
\[ \hat{p}_{k+1}(0) = 1 \]

minimal polynomial of \( v \):
minimal degree of all monic polynomials \( p(t) \) \( (t^k + \ldots) \) such that
\[ p(A)v = 0 \]
(called grade of \( v \))
remember \( p_\alpha (A) = 0 \) where \( p_\alpha \) = characteristic polynomial
grade \( \leq n \)
\[ (A - \lambda I) v = 0 \implies \text{grade} = 1. \]

\[ \hat{H}_4 = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix} \]
\( \text{size} 5 \times 4 \) \( (m+1) \times m \)
Hessenberg form = quasi-triangular

-----------------------------------------------
At end of loop

\[ w = A v_j - h_{1j} v_1 - h_{2j} v_2 \ldots - h_{jj} v_j \]

w is the w before scaling - After scaling:

\[ v_{j+1} = w / h_{j+1,j} \quad \text{with} \quad h_{j+1,j} = ||w|| \]

\[ w = h_{j+1,j} v_{j+1} \]

\[ A v_j = w + \sum_{i \leq j} h_{ij} v_i = h_{j+1,j} v_{j+1} + \sum_{i \leq j} h_{ij} v_i \]

\[ A v_j = \sum_{i=1}^{j+1} h_{ij} v_i \]

Write this for all columns \( j = 1, : m \)

\[ \text{--------------------------} \]

\[ A V_m = V_{m+1} H_m \]

\[ \text{--------------------------} \]

\[ H_m = \text{Hessensberg matrix of size } (m+1)xm \]

\[ [H_m] \]

\[ A V_m = V_{m+1} H_m = [V_m, v_{m+1}] \]

\[ = V_m H_m + v_{m+1} h^T \]

\[ V^T A V_m = H_m \quad \text{mxm Hessenberg} \]

\[ \text{=================================} \]

Arnoldi's method for linear system (FOM)

projection method with

\( K = K_m, L = K_m \)

We have a certain \( x_0 \) \( \Rightarrow r_0 = b - A x_0 \)

take \( v_1 = r_0 / ||r_0|| \)

\[ \ddot{x} = x_0 + V_m y \quad (y \in \mathbb{R}^n) \]

want \( \ddot{r} = b - A \ddot{x} = r_0 - A V_m y \perp K_m \Rightarrow \)

\[ V_m^T(b - A \ddot{x}) = 0 \Rightarrow V_m^T [r_0 - A V_m y] = 0 \Rightarrow \]

\[ V_m^T r_0 - H_m y = 0 \]

Let \( \beta = ||r_0|| \) \( \Rightarrow r_0 = \beta v_1 \Rightarrow V_m^T r_0 = \beta e_1 \]
\( \beta e_1 - H y = 0 \)

approx. sol. =

============================
\( x_n = x_0 + V y \) where \( y = H^{-1} (\beta e_1) \)

============================