A Geometric Approach to Robustness in Complex Networks

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***Abstract*—We explore the geometry of networks in terms of an n-dimensional Euclidean embedding represented by the Moore-Penrose pseudo-inverse of the graph Laplacian $(\mathbf{L}^+)$. The reciprocal of squared distance from each node i to the origin in this n-dimensional space yields a structural centrality index $(\mathcal{C}^*(i))$ for the node; while the harmonic sum of individual node structural centrality indices, $\sum_i 1/\mathcal{C}^*(i)$, i.e. the trace of $\mathbf{L}^+$, yields the well-known Kirchoff index $(\mathcal{K})$, an overall structural descriptor for the network. In addition to its geometric interpretation, we provide alternative interpretation of the proposed structural centrality index $(\mathcal{C}^*(i))$ of each node in terms of forced detour costs and recurrences in random walks and electrical networks. Through empirical evaluation over example and real world networks, we demonstrate how structural centrality is better able to distinguish nodes in terms of their structural roles in the network and, along with Kirchoff index, is appropriately sensitive to perturbations/rewirings in the network.**

I. Introduction

Unlike traditional studies on network robustness, that typically treat networks as combinatoric objects and rely primarily on classical graph-theoretic concepts (e.g. minimum cuts) to characterize network robustness, we explore a geometric approach which enables us to employ more advanced theories and techniques, quantify and compare robustness of networks in terms of their local and global structures.

In this work, we study a geometric embedding of networks using the Moore-Penrose (pseudo) inverse of the graph Laplacian for the network, denoted henceforth by $\mathbf{L}^+$. We show that the diagonal entries of $\mathbf{L}^+$, that represent the squared distance of each node to the origin in the $n - dimensional$ Euclidean space of the network embedding, provide a robust structural centrality measure $(\mathcal{C}^*(i) = 1/l^+_{ii})$ for the nodes in the network. In particular, closer a node i is to the origin in this space, more structurally central it is. Moreover, the trace of $\mathbf{L}^+$, $Tr(\mathbf{L}^+)$, also called the *Kirchoff index* $(\mathcal{K})$, provides a structural robustness index for the network as a whole. Once again, lower the value of $\mathcal{K}$ for a network, more compact the embedding, and more structurally robust the overall network is.

In addition to the geometric interpretation of structural centrality and Kirchoff index, as described above, we provide two alternative, albeit related, interpretations as well. First, we equate structural centrality of a node i, i.e. $\mathcal{C}^*(i)$, to the (reciprocal of) average detour overhead incurred when a random walk between any source destination pair of nodes is forced to go through node i. Intuitively, the average overhead incurred in such detours (measured in terms of the number of steps in the random walk) will be higher for structurally peripheral nodes (relatively lower $\mathcal{C}^*(i)$ and higher l^+_{ii}) as compared to structurally central ones (relatively higher $\mathcal{C}^*(i)$ and lower l^+_{ii}). Secondly, we show how $\mathcal{C}^*(i)$ is related to electrical voltages when the network is treated as an equivalent electrical network (EEN). This, in turn, yields an interpretation of $\mathcal{C}^*(i)$ in terms of the probability with which a random detour through i returns to the source node (also referred to as the phenomenon of *recurrence* in random walk literature). More precisely, higher $\mathcal{C}^*(i)$ means a random detour through node i forces the random walk to return to the source node with lower probability, thereby incurring lower detour overhead.

Through numerical simulations using synthetic and realistic network topologies, we demonstrate that our new indices better characterize robustness of nodes in network as compared to other existing metrics (e.g. node centrality measured based on degree, shortest paths, etc.). A rank-order of nodes in terms of their structural centralities helps distinguish them in terms of their structural roles (such as core, gateway, etc.). Also, structural centrality and the Kirchoff index, are both appropriately sensitive to local perturbations in the network, a property not displayed by other centralities in literature (as shown later in this paper).

The rest of the paper is organized as follows: We begin by providing a brief overview of the literature and the several structural indices that have been proposed in §II. §III describes a geometric embedding of the network using the eigen-space of $\mathbf{L}^+$ and introduce structural centrality and Kirchoff index as measures of robustness §IV demonstrates how structural centrality of a node reflects the average detour overhead in random walks through a particular node in question, §V presents comparative empirical analysis and in §VI the paper is concluded.

II. Related Work

Robustness of nodes to failures in complex networks is dependent on their overall *connectedness* in the network. Several centralities, that characterize *connectedness* of nodes in complex networks in varying ways, have therefore been proposed in literature. Perhaps the simplest of all is degree — the number of edges incident on a node. Except in *scale free* networks that display *rich club connectivity* [2], [7], [8], degree is essentially a *local* measure (a first oder/one-hop connectedness index) and does not determine the overall connectedness of a node. A similar metric based on *joint-degree*, i.e. the product of degrees of a pair of nodes that are directly connected to each other in the network through an edge, is a second-order measure of connectedness and is equally limited.

A more sophisticated measure of centrality is geodesic closeness (GC) [10], [11]. It is defined as the (reciprocal of) average shortest-path distance of a node from all other nodes in the network. Clearly, geodesic closeness is a p^{th}-order measure where $p = \{1, 2, ..., \delta\}$, δ being the geodesic diameter of the graph, and is better for characterizing global connectedness properties than either degree or joint-degree of nodes. However, communication in networks is not always confined to shortest paths alone and geodesic based indices, that ignore other alternative paths between nodes, only partially capture connectedness of nodes.

Recently, subgraph centrality (SC) — the number of subgraphs of a graph that a node participates in — has also been proposed [6]. In principle, a node with high subgraph centrality, should be better connected to other nodes in the network through redundant paths. Alas, subgraph centrality is computationally intractable and the proposed index in [6] approximates subgraph centrality by the sum of lengths of all *closed* walks, weighed in inverse proportions by the factorial of their lengths; which inevitably introduces local connectivity bias.

Our aim in this work, therefore, is to provide an index for robustness of nodes in a given network, as a comparative measure between any pair of nodes, as well as one for the overall network that effectively reflect global connectedness properties.

III. Geometric Embedding of Networks using $\mathbf{L}^+$ and Structural Centrality

In studying the *geometry* of networks, we first need to embed a network (e.g. represented abstractly as a graph) into an appropriate geometric space endowed with a metric function (mathematically, a metric space). In this section we describe an n-dimensional embedding of the complex network using, the Moore-Penrose pseudo-inverse of the combinatorial laplacian $(\mathbf{L}^+)$. The squared length of the position vector for a node in this space yields a geometric measure of centrality for the node while the sum of the squared lengths of the position vectors of all nodes, or the trace of $\mathbf{L}^+$, yields an overall robustness index for the graph. But first we need to introduce some basic notations.

Given a complex network, its topology is in general represented as a (weighted) graph, $G = (V, E, W)$, where $V(G)$ is the set of nodes representing, say, switches, routers or end systems in the network; $E = \{e_{uv} : u, v \in V\}$ is the set of edges connecting pairs of nodes representing, for example, the (physical or logical) communication links between the pair of nodes; and $W = w_{uv} \in \Re^+ : e_{uv} \in E(G)$ is a set of weights assigned to each edge of the graph (here $\Re^+$ denotes the set of nonnegative real numbers). These weights can be used to represent, for example, the capacity, latency, or geographical distance, or an (administrative) routing cost associated with the edge (communication link) e_{uv}. Note that if w_{uv} is simply 0 or 1, we have a simple and unweighted graph.

Given $G = (V, E, W)$, we introduce an $n \times n$ *affinity* matrix $\mathbf{A} = [a_{ij}]$ associated with G, where $n = |V(G)|$ is the number of nodes in G (the *order* of G), and $a_{ij} \geq 0$ is some function of the weight w_{ij}. For a simple graph where $w_{ij} \in \{0, 1\}$, setting $a_{ij} = w_{ij}$ yields the standard adjacency matrix of the graph G. In general, each entry a_{ij} captures some measure of affinity between nodes i and j: the larger a_{ij} is, nodes i and j are in a sense *closer* or more *strongly connected*. Hence in general, we refer to $\mathbf{A}$ as an affinity matrix associated with G. We assume that $a_{ij} = a_{ji}$, i.e. $\mathbf{A}$ is symmetric. For $1 \leq i \leq n$, define $d(i) = \sum_j a_{ij}$, and refer to $d(i)$ as the (generalized) degree of node i. (Note that if G is a simple unweighted graph j and $\mathbf{A}$ is its adjacency matrix, then $d(i)$ is the degree of node i.)

The *combinatorial Laplacian* of $\mathbf{A}$ (or the associated graph G), is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D} = [d_{ii}] = d(i)$ is a diagonal matrix with $d(i)$'s on the diagonal. The Laplacian is a positive semidefinite matrix, and thus has n non-negative Eigen values λ_i's. For $1 \leq i \leq n$, let $\mathbf{u}_i$ be the corresponding eigenvector of λ_i such that

$||\mathbf{u}_i||_2^2 = \mathbf{u}_i'\mathbf{u}_i$. We assume that the eigenvalues λ_i's are ordered such that $\lambda_1 \geq ... \geq \lambda_n = 0$. Then the matrix formed by the corresponding eigenvectors $\mathbf{u}_i's$, $\mathbf{U} = [\mathbf{u}_1, ..., \mathbf{u}_n]$, is orthogonal i.e. $\mathbf{U}'\mathbf{U} = \mathbf{I}$, the identity matrix. More importantly, $\mathbf{L}$ admits an eigen decomposition $\mathbf{L} = \mathbf{U}\Lambda\mathbf{U}'$, where Λ is the diagonal matrix $\Lambda = [\lambda_{ii}] = \lambda_i$.

Like $\mathbf{L}$, its Moore-Penrose (pseudo) inverse $\mathbf{L}^+$ is also positive semi-definite, and admits an eigen decomposition of the form, $\mathbf{L}^+ = \mathbf{U}'\Lambda^+\mathbf{U}$, where $\mathbf{\Lambda}^+$ is a diagonal matrix consisting of λ^{-1} if $\lambda_i > 0$, and 0 if $\lambda_i = 0$ (for simplicity of notation, in the following we will use the convention $\lambda_i^{-1} = 0$ if $\lambda_i = 0$). Define $\mathbf{X} = \Lambda^{+1/2}\mathbf{U}$. Hence, $\mathbf{L}^+ = \mathbf{X}'\mathbf{X}$ which means that the network can be embedded into the Euclidean space $\Re^n$ where the coordinates of node i are given by $\mathbf{x}_i$, the i^{th} column of $\mathbf{X}$. As the centroid of the position vectors lies at the origin in this n-dimensional space [9], the squared distance of node i from the origin is exactly the corresponding diagonal entry of $\mathbf{L}^+$ i.e. $||\mathbf{x}_i||_2^2 = l_{ii}^+$ and the squared distance between two nodes $i, j \in V(G), ||\mathbf{x}_i - \mathbf{x}_j||_2^2 = Vol(G)^{-1}C_{ij}$ where $Vol(G) = \sum_{i=1}^n d(i)$ is called the *volume* of the graph (a constant for the graph) and C_{ij} is called the *commute time* defined as the expected length of commute in a random walk between i and j in the network [4].

Based on the geometric embedding of the graph using $\mathbf{L}^+$ described above, we now put forth two new robustness metrics. First, a rank order for individual nodes in terms of their relative robustness properties called *structural centrality*, defined as:

$$\mathcal{C}^*(i) = 1/l_{ii}^+, \quad \forall i \in V(G) \tag{1}$$

Specifically, closer a node is to the origin in this n-dimensional space, more structurally central it is and vice versa. Next, a structural descriptor for the overall robustness of the network called *Kirchoff index*, defined as:

$$\mathcal{K}(G) = Tr(\mathbf{L}^+) = \sum_{i=1}^n l_{ii}^+ = \sum_{i=1}^n 1/\mathcal{C}^*(i) \tag{2}$$

Geometrically, more compact the embedding is, or equivalently lower the value of $\mathcal{K}(G)$, more robust the network G is. We can therefore use Kirchoff index to compare the robustness of two graphs with the same order and volume.

In what follows, we demonstrate how these two metrics indeed reflect robustness of nodes and the overall graph respectively, first through rigorous mathematical analysis and then with empirical evaluations.

IV. Structural Centrality, Random Detour Costs and Electrical Voltages

To show that structural centrality ($\mathcal{C}^*$) and Kirchoff index ($\mathcal{K}$) indeed provide a measures of robustness, we relate them to the lengths of random walks on the graph. In §IV-A, we demonstrate how $\mathcal{C}^*(k)$ for node captures an overhead in random *detours* through node k as a *transit* vertex. Next in §IV-B, we provide an electrical interpretation for the same.

A. Detours in Random Walks

A simple random walk $(i \to j)$, is a discrete stochastic process that starts at a node i, the source, visits other nodes in the graph G and stops on reaching the destination j [12]. In contrast, we define a *random detour* as:

Definition 1: Random Detour $(i \to k \to j)$: A random walk starting from a source node i, that must visit a transit node k, before it reaches the destination j and stops.

Effectively, such a random detour is a combination of two simple random walks: $(i \to k)$ followed by $(k \to j)$. We quantify the difference between the random detour $(i \to k \to j)$ and the simple random walk $(i \to j)$ in terms of the number of steps required to complete each of the two processes given by hitting time.

Definition 2: Hitting Time (H_{ij}): The expected number of steps in a random walk starting at node i before it reaches node j for the first time.

Clearly, $H_{ik} + H_{kj}$ is the expected number of steps in the random detour $(i \to k \to j)$. Therefore, the overhead incurred is:

$$\Delta H^{i \to k \to j} = H_{ik} + H_{kj} - H_{ij} \tag{3}$$

Intuitively, more peripheral transit k is, greater the overhead in (3). The overall peripherality of k is captured by the following average:

$$\Delta H^{(k)} = \frac{1}{n^2 \; Vol(G)} \sum_{i=1}^n \sum_{j=1}^n \Delta H^{i \to k \to j} \tag{4}$$

Alas, hitting time is not a Euclidean distance as $H_{ij} \neq H_{ji}$ in general. An alternative is to use commute time $C_{ij} = H_{ij} + H_{ji} = C_{ji}$, a metric, instead. More importantly [14],

$$C_{ij} = Vol(G)(l_{ii}^+ + l_{jj}^+ - l_{ij}^+ - l_{ji}^+) \tag{5}$$

and in the overhead form (3), (non-metric) hitting and (metric) commute times are in fact equivalent (see propositions $9-58$ in [13] and Theorem 1 in [18]):

$$\Delta H^{i \to k \to j} = (C_{ik} + C_{kj} - C_{ij})/2 = \Delta H^{j \to k \to i} \tag{6}$$

We now exploit this equivalence to equate the cumulative detour overhead through transit k from (4) to l^+_{kk} in the following theorem.

Theorem 1: $\Delta H^{(k)} = l^+_{kk}$

Proof: Using $\Delta H^{i \to k \to j} = (C_{ik} + C_{kj} - C_{ij})/2$:

$$\Delta H^{(k)} = \frac{1}{2n^2 \ Vol(G)} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ik} + C_{kj} - C_{ij}$$

Observing $C_{xy} = Vol(G) \ (l^+_{xx} + l^+_{yy} - 2l^+_{xy})$ [14] and that $\mathbf{L}^+$ is doubly centered (all rows and columns sum to 0) [9], we obtain the proof.
□

Therefore, a low value of $\Delta H^{(k)}$ implies higher $\mathcal{C}^*(k)$ and more structurally central node k is in the network. Theorem 1 is interesting for several reasons. First and foremost, note that:

$$\sum_{j=1}^{n} C_{kj} = Vol(G) \ (n \ l^+_{kk} + Tr(\mathbf{L}^+)) \quad (7)$$

As $Tr(\mathbf{L}^+)$ is a constant for a given graph and an invariant with respect to the set $V(G)$, we obtain $l^+_{kk} \propto \sum_{j=1}^{n} C_{kj}$; lower l^+_{kk} or equivalently higher $\mathcal{C}^*(k)$, implies shorter average commute times between k and the rest of the nodes in the graph on an average. It is well understood that low C_{kj} reflects greater number of alternative (redundant) paths between nodes k and j; which in turn shows better connectivity between the two nodes [4]. Therefore, lower the value of $\mathcal{C}^*(k)$, greater the number of redundant paths between the node k and the rest of the network and consequently more immune is node k to random failures in the network. Moreover,

$$\mathcal{K}(G) = Tr(\mathbf{L}^+) = \sum_{k=1}^{n} l^+_{kk} = \frac{1}{2nVol(G)} \sum_{k=1}^{n} \sum_{j=1}^{n} C_{kj} \quad (8)$$

As $\mathcal{K}(G)$ reflects the average commute time between any pair of nodes in the network, it is a measure of overall structural robustness of G. For two networks of the same order (n) and volume $(Vol(G))$, the one with lower $\mathcal{K}(G)$ has a greater number of redundant paths between any pair of nodes in the network and hence is more immune to random edge failures.

B. Recurrence, Voltage and Electrical Networks

Interestingly, the detour overhead in (3) is related to *recurrence* in random walks — the expected number of times a random walk $(i \to j)$ returns to the source i [5]. We now explore how recurrence in detours related to structural centrality of nodes. But first we need to introduce some terminology.

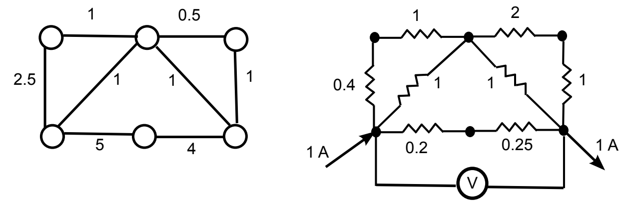


Fig. 1. A simple graph G and its EEN.

The equivalent electrical network (EEN) [5] for $G(V, E, W)$ is formed by replacing an edge $e_{ij} \in E(G)$ with a resistance equal to w_{ij}^{-1} (see Fig. 1). The *effective resistance* (Ω_{ij}) is defined as the voltage developed across a pair of terminals i and j when a unit current is injected at i and is extracted from j, or vice versa. In the EEN, let V_k^{ij} be the voltage of node k when a unit current is injected at i and a unit current is extracted from j. From [19], $U_k^{ij} = d(k)V_k^{ij}$. Substituting $k = i$ we get, $U_i^{ij} = d(i)V_i^{ij}$; the expected number of times a random walk $(i \to j)$ returns to the source i. For a finite graph G, $U_i^{ij} > 0$. The following theorem connects recurrence to the detour overhead.

Theorem 2:

$$\Delta H^{i \to k \to j} = V_i^{ik} + V_i^{kj} - V_i^{ij} = \frac{Vol(G) \ (U_i^{ik} + U_i^{kj} - U_i^{ij})}{d(i)}$$

Proof: From [19] we have, $\Delta H^{i \to k \to j} = d(i)^{-1} \ Vol(G) \ U_i^{jk}$. The rest of this proof follows by proving $U_i^{jk} = U_i^{ik} + U_i^{kj} - U_i^{ij}$.

From the *superposition principle* of electrical current, we have $V_x^{xz} = V_y^{xz} + V_y^{zx}$. Therefore,

$$V_i^{ik} + V_i^{kj} - V_i^{ij} = V_j^{ik} + V_j^{ki} + V_i^{kj} - V_k^{ij} + V_k^{ji} = V_j^{ik} + (V_j^{ki} + V_i^{kj} - V_k^{ij} - V_k^{ji})$$

From the *reciprocity principle*, $V_z^{xy} = V_x^{zy}$. Therefore, $V_i^{ik} + V_i^{kj} - V_i^{ij} = V_i^{jk}$. Multiplying by $d(i)$ on both sides we obtain the proof.
□

The term $(U_i^{ik} + U_i^{kj}) - U_i^{ij}$ can be interpreted as the expected extra number of times a random walk returns to the source i in the random detour $(i \to k \to j)$ as compared to the simple random walk $(i \to j)$. Each instance of the random process that returns to the source, must effectively start all over again. Therefore, more often the walk returns to the source greater the expected number of steps required to complete the process and less central the transit k is, with respect to the source-destination pair (i, j).

Therefore, $\Delta H^{(k)}$, that is the average of $\Delta H^{i \to k \to j}$ over all source destination pairs, tells us the average increase in recurrence caused by node k in random detours between any source destination pair in the network. Higher the increase in recurrence, i.e. $(\Delta H^{(k)})$, less structurally central the node k is in the network.

V. Empirical Evaluations

We now empirically study the properties of structural centrality $(\mathcal{C}^*(i))$ and Kirchoff index (we use $\mathcal{K}^* = \mathcal{K}^{-1}$ henceforth to maintain *higher is better*). We first show in §V-A, how structural centrality can capture the structural roles played by nodes in the network and then in §V-B demonstrate how it, along with Kirchoff index, is appropriately sensitivity to rewiring and local perturbations in the network.

A. Identifying Structural Roles of Nodes

Consider the router level topology of the Abilene network (Fig. 2(a)) [1]. At the core of this topology, is a ring of 11 POP's, spread across mainland US, through which several networks interconnect. Clearly, the connectedness of such a network is dependent heavily on the low degree nodes on the ring. For illustration, we mimic the Abilene topology, with a simulated network (Fig. 2(b)) which has a 4-node core $\{v_1, ..., v_4\}$ that connects 10 networks through gateway nodes $\{v_5, ..., v_{14}\}$ (Fig.2(b)). Fig. 3 shows the (max-normalized) values of GC, SC and $\mathcal{C}^*$ for the core $\{v_1, ..., v_4\}$, gateway $\{v_5, ..., v_{14}\}$ and other nodes $\{v_{15}, ..., v_{65}\}$ in topology (Fig.2(b)). Notice that v_5 and v_6 are the highest degree nodes $(d(v_5) = d(v_6) = 10)$ in the network while v_{14} has the highest subgraph centrality (SC). In contrast, $\mathcal{C}^*$ ranks the core nodes higher than the gateway nodes with v_1 at the top. The relative peripherality of v_5, v_6 and v_{14} as compared to the core nodes requires no elaboration. As far as geodesic centrality (GC) is concerned, it ranks all the nodes in the subnetwork abstracted by v_5, namely $v_{15} - v_{23}$, as equally well connected even though v_{22} and v_{23} have redundant connectivity to the network through each other and are, ever so slightly, better connected than the others.

We see similar characterization of structural roles of nodes in two real world networks: western states power-grid network [20] and a social network of co-authorships [16], as shown through a color scheme based on $\mathcal{C}^*(i)$ values in Fig. 4. Core-nodes connecting different sub-communities of nodes in both these real world networks are recognized effectively by structural centrality as being more central (Red end of the spectrum) than several higher degree peripheral nodes.

B. Sensitivity to Local Perturbations

An important property of centrality measures is their sensitivity to perturbations in network structure. Traditionally, structural properties in real world networks have been equated to average statistical properties like power-law/scale-free degree distributions and rich club connectivity [2], [7], [8]. However, the same degree sequence $D = \{d(1) \geq d(2) \geq ... \geq d(n)\}$, can result in graphs of significantly varying topologies. Let $\mathcal{G}(D)$ be the set of all connected graphs with scaling sequence D. The generalized Randic index $R_1(G)$ [3], [17]:

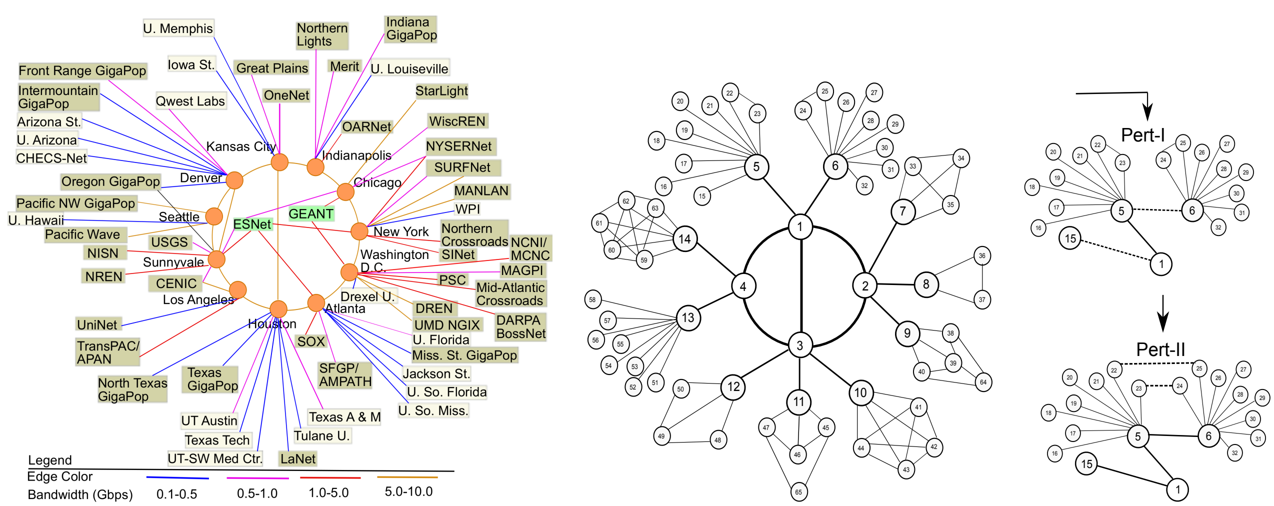
$$R_1(G) = \sum_{e_{ij} \in E(G)} d(i)d(j) \tag{9}$$

where $G \in \mathcal{G}(D)$, is considered to be a measure of overall connectedness of G as higher $R_1(G)$ suggests *rich club connectivity* (RCC) in G [15]. Also, the average of each centrality index (degree, GC, SC averaged over the set of nodes), is in itself a global structural descriptor for the graph G [6]. We now examine the sensitivity of each index with respect to local perturbations in the subnetwork abstracted by the core node v_1 and its two gateway neighbors v_5 and v_6.

First, we rewire edges $e_{15,5}$ and $e_{6,1}$ to $e_{15,1}$ and $e_{6,5}$ respectively (PERT-I Fig. 2(c)). PERT-I is a degree preserving rewiring which only alters local connectivities. Fig. 5(a) shows the altered values of centralities $(\mathcal{C}^*, GC, SC)$ after PERT-I. Note, after PERT-I, v_{15} is directly connected to v_1 which makes $\mathcal{C}^*(v_{15})$ comparable to other gateway nodes while the value of $SC(v_{15})$, relative to the other gateway nodes, seems to be entirely unaffected. Moreover, PERT-I also results in v_6 losing its direct link to the core, reflected in the decrease in $\mathcal{C}^*(v_6)$ and a corresponding increase in $\mathcal{C}^*(v_5)$. $\mathcal{C}^*$, however, still ranks the core nodes higher than v_5 (whereas SC does not) because PERT-I being a local perturbation should not affect nodes outside the sub-network — v_1 continues to abstract the same sub-networks from the rest of the topology. We, therefore, observe that $\mathcal{C}^*$ is appropriately sensitive to the changes in connectedness of nodes in the event of local perturbations. But what about the network on a whole? Let G and G_1 be the topologies before and after PERT-I. G_1 is less well connected overall than G as the failure of $e_{5,1}$ in G_1 disconnects 19 nodes from the rest of the network as compared to 10 nodes in G. However,

$$\Delta R_1(G \to G_1) = \frac{R_1(G_1) - R_1(G)}{R_1(G)} = 0.029$$

as the two highest degree nodes (v_5 and v_6) are directly connected in G_1 (see Tab. I for the sensitivity of other



(a) Abilene Topology (b) Simulated topology (c) Perturbations

Fig. 2. Abilene Network and a simulated topology.

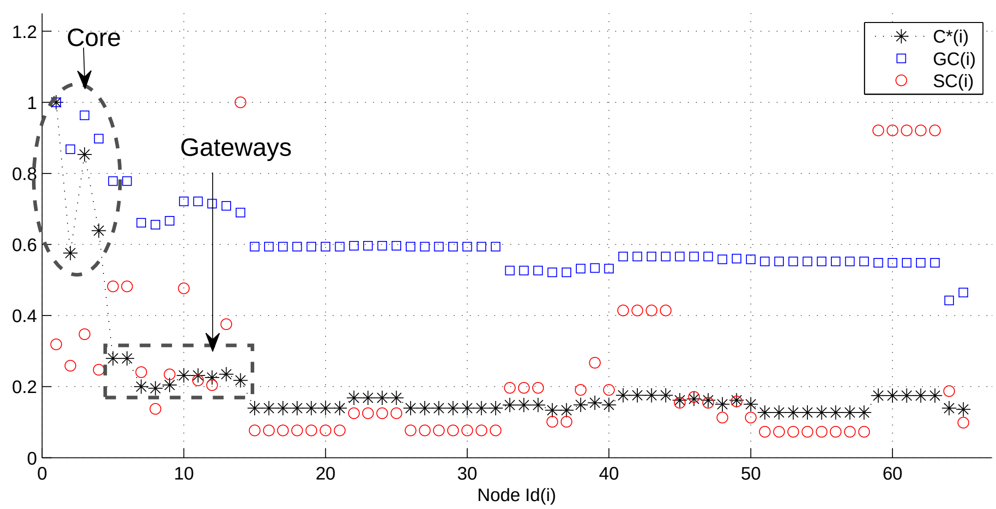


Fig. 3. Max-normalized centralities for simulated topology.

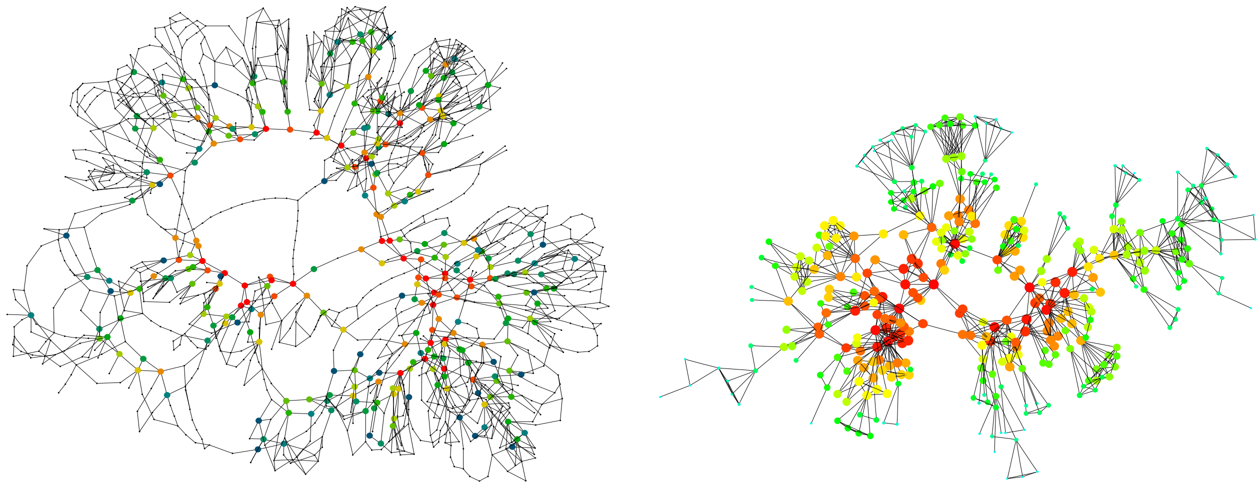
TABLE I

SENSITIVITY OF STRUCTURAL DESCRIPTORS TO LOCAL PERTURBATIONS.

#	Structural Descriptor	PERT-I	PERT-II
1	$\mathcal{K}^*(G)$ (or $\sum_i^n \mathcal{C}^*(i)$)	$\downarrow$	$\uparrow$
3	$\sum_i^n d(i)$	$\leftrightarrow$	$\leftrightarrow$
3	$R_1(G)$	$\uparrow$	$\leftrightarrow$
4	$\sum_i^n GC(i)$	$\downarrow$	$\uparrow$
5	$\sum_i^n SC(i)$	$\uparrow$	$\downarrow$

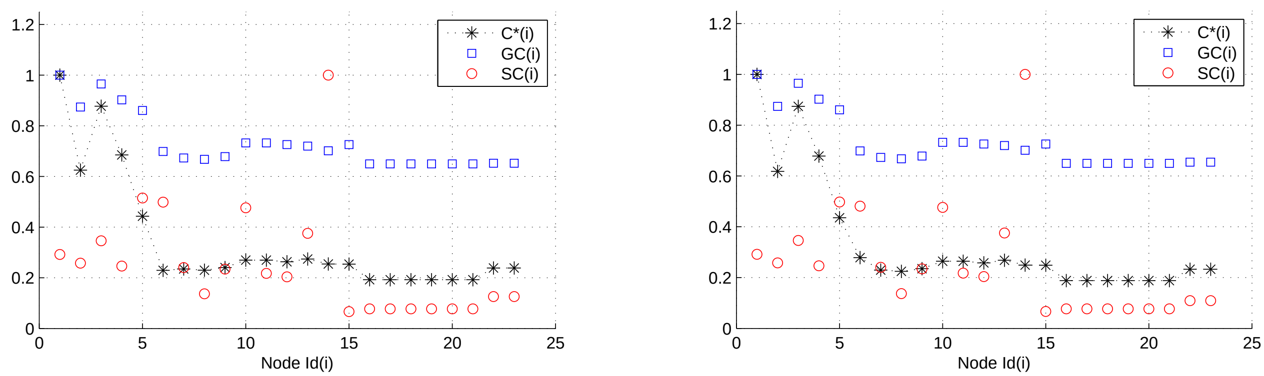
centrality based global structural descriptors). In contrast, $\Delta\mathcal{K}^*(G \to G_1) = -0.045$, which rightly reflects the depreciation in overall connectedness after PERT-I.

A subsequent degree preserving perturbation PERT-II of G_1, rewiring $e_{22,23}$ and $e_{24,25}$ to $e_{22,25}$ and $e_{23,24}$, to obtain G_2, creates two cycles in G_2 that safeguard against the failure of edge $e_{5,6}$. This significantly improves local connectivities in the sub-network. However, $\Delta R_1(G_1 \to G_2) = 0$ (and average SC decreases) while $\Delta\mathcal{K}^*(G_1 \to G_2) = 0.036$ which once again shows the efficacy of Kirchoff index as a measure of global connectedness of networks.



(a) The western-states power grid network [20] (b) A network of co-authorships in network sciences [16]

Fig. 4. Real world networks: $Red \rightarrow Turqoise$ reducing order of $\mathcal{C}^*$



(a) After PERT-I (b) After PERT-II

Fig. 5. Max-normalized values of structural centrality, geodesic closeness and subgraph centrality for core, gateway and some other nodes.

VI. Conclusion and Future Work

In this work, we presented a geometric perspective on robustness in complex networks. We proposed structural centrality $\mathcal{C}^*(i)$ and Kirchoff index $\mathcal{K}$ respectively as measures of robustness of individual nodes and the overall network. Additionally, we provided interpretations for these indices in terms of the overhead incurred in random detours through a node in question as well as in terms of the recurrence probabilities and voltage distribution in the EEN corresponding to the network. Both indices reflect, in some sense, the number of redundant/alternative paths in the network thereby capturing global connectedness. Through numerical analysis on simulated and real world networks, we demonstrated that $\mathcal{C}^*(i)$ captures structural roles played by nodes in networks and, along with Kirchoff index, is suitably sensitive to perturbations/rewirings in the network. In future, we aim at investigating similar metrics for the case of strongly connected weighted directed graphs to further generalize our work.

VII. Acknowledgment

This work was supported in part by the NSF grants CNS-0905037, CNS-1017647 and CNS-1017092 and the DTRA grant HDTRA1-09-1-0050.

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