

Support Envelopes: A Technique for Exploring the Structure of Association Patterns

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ABSTRACT

This paper introduces support envelopes—a new tool for analyzing association patterns—and illustrates some of their properties, applications, and possible extensions. Specifically, the *support envelope* for a transaction data set and a specified pair of positive integers (m, n) consists of the items and transactions that need to be searched to find any association pattern involving m or more transactions and n or more items. For any transaction data set with M transactions and N items, there is a unique lattice of at most $M * N$ support envelopes that captures the structure of the association patterns in that data set. Because support envelopes are not encumbered by a support threshold, this *support lattice* provides a complete view of the association structure of the data set, including association patterns that have low support. Furthermore, the boundary of the support lattice—the *support boundary*—has at most $\min(M, N)$ envelopes and is especially interesting since it bounds the maximum sizes of potential association patterns—not only for frequent, closed, and maximal itemsets, but also for patterns, such as error-tolerant itemsets, that are more general. The association structure can be represented graphically as a two-dimensional scatter plot of the (m, n) values associated with the support envelopes of the data set, a feature that is useful in the exploratory analysis of association patterns. Finally, the algorithm to compute support envelopes is simple and computationally efficient, and it is straightforward to parallelize the process of finding all the support envelopes.

Categories and Subject Descriptors: H.2.8 [Database Management]: Database Applications - Data Mining

General Terms: Algorithms, Theory

Keywords: support envelope, association analysis, formal concept analysis, error-tolerant itemsets

1. INTRODUCTION

This paper introduces *support envelopes*, which provide a compact and computationally efficient approach for cap-

turing the high-level structure of support based association patterns such as frequent, closed, and error-tolerant itemsets [8, 9, 13, 20, 19]. More specifically, the *support envelope* for a transaction data set and a pair of positive integers (m, n) consists of the items and transactions that need to be searched to find all patterns involving a minimum of m transactions and n items. Thus, the number of distinct envelopes is at most $M * N$, where M is the number of transactions and N is the number of items. For sparse data sets, the number of support envelopes is typically much less than $M * N$.

Like other association patterns, e.g., frequent and closed itemsets, support envelopes form a lattice. (We will call this lattice the *support lattice*.) In particular, for any binary matrix with M rows (transactions) and N columns (items), there is a unique lattice that captures the structure of the association patterns for that matrix. Standard association analysis techniques for finding frequent, closed, or maximal itemsets break down when the support threshold becomes too low, and thus, cannot provide information about patterns with low levels of support. However, because support envelopes are not encumbered by a support threshold, the support lattice provides a complete view of the association structure of the data set, including association patterns that have low support.

The boundary (positive border [11]) of the support lattice, which we call the *support boundary*, is especially interesting since it bounds the maximum sizes of potential association patterns. For example, if an envelope on the support boundary is characterized by the pair of integers (m, n) , then there are no non-empty envelopes or association patterns with larger values of m and n . To illustrate, if the support boundary of a data set contains the support envelope characterized by $(10, 5)$, then the data set cannot contain a frequent itemset with 5 or more items that also has more than 10 supporting transactions. Furthermore, the support boundary is not just a bound on frequent, closed, and maximal itemsets, but also on patterns, such as error-tolerant itemsets (ETIs) [19], that are more general. Another attractive property of the support boundary is that it contains at most $\min(M, N)$ support envelopes.

In addition to providing us with a theoretical foundation for understanding—at least in part—the high-level structure of association patterns, support envelopes allow the association structure of a binary data to be represented graphically. Specifically, the (m, n) values associated with the envelopes in the data set can be displayed in a two-dimensional scatter plot, which provides a global view of the structure of associ-

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ation patterns in a data set. Additional information can be added to this scatter plot by shading or coloring each point based on the density—fraction of ones—of its corresponding support envelope. Support envelopes are often fairly sparse, i.e., have low density, but in some cases, support envelopes can be relatively dense, and hence, are interesting not just because of the information that they provide about other association patterns, but also as patterns in their own right. Thus, for some data sets, e.g., the kosarak data set discussed in Section 4, we can use the scatter plot to identify dense envelopes that can be investigated to find association patterns of considerable interest, e.g., low support itemsets with many items.

Support envelopes are also interesting from an algorithmic point of view. We present a simple iterative algorithm that, given only (m, n) and a binary data matrix, finds the support envelope for all the support-based patterns characterized by (m, n) in time proportional to (a) nnz , the number of non-zero entries in the data matrix, and (b) num_iter , the number of iterations, which is typically small, i.e., less than 10 for data sets in this paper. This algorithm can compute any individual support envelope without using the preceding elements of the support lattice. (As discussed later, using the lattice does, however, improve efficiency somewhat, i.e., the number of iterations is typically reduced to 2 or 3.) The computational complexity of computing the entire support lattice is $O(M * N * nnz * num_iter)$, while the computational complexity of finding only the support boundary is $O(\min(N \log(M), M \log(N)) * nnz * num_iter)$. Thus, unlike many association mining tasks, the computational complexity of finding the support lattice or support boundary is not inherently exponential. Also, because support envelopes can be computed independently of the lattice, it should be straightforward to parallelize finding either all envelopes or just those envelopes on the support boundary.

Finally, we consider two simple extensions of support envelopes. First, support envelopes involving only the items of one specific transaction provide a view of patterns with respect to a particular transaction, i.e., the set of such ‘restricted’ support envelopes will involve only those association patterns that are guaranteed to contain the specified transaction. Second, we consider support envelopes that involve an additional constraint on items, namely, that a certain fraction of the occurrences of an item must be in the support envelope. Such an approach eliminates items that are frequent across many patterns.

Overview Section 2 introduces support envelopes and an algorithm for finding them via an informal example, while Section 3 provides a formal analysis of support envelopes and their properties, and also discusses algorithms for finding an individual support envelope, the envelopes on the support boundary, and the set of all envelopes for a data set. Section 4 uses support envelopes to explore real transaction data sets, while extensions to the support envelope concept are considered in Section 5. Section 6 discusses related work, and we conclude, in Section 7, with a summary and indications for future work.

2. SUPPORT ENVELOPES: AN INFORMAL INTRODUCTION

This section contains an informal introduction to support envelopes, lattices, and boundaries. Consider the following task: Construct an algorithm to identify all the items and

Table 1: Original transaction data set.

	A	B	C	D	E	row sum
1	1	0	1	1	1	4
2	0	1	0	1	0	2
3	0	1	1	1	1	4
4	0	0	1	0	1	2
5	0	1	0	1	0	2
6	0	1	0	1	0	2
7	1	0	1	1	1	4
8	1	0	1	1	1	4
9	1	0	0	1	0	2
10	1	0	1	1	1	4
11	0	1	1	1	0	3
12	1	0	0	0	1	2
col sum	6	5	7	10	7	

Table 2: Data after eliminating rows with less than 3 items. Step 1.

	A	B	C	D	E	row sum
1	1	0	1	1	1	4
3	0	1	1	1	1	4
7	1	0	1	1	1	4
8	1	0	1	1	1	4
10	1	0	1	1	1	4
11	0	1	1	1	0	3
col sum	4	2	6	6	5	

Table 3: Data after eliminating columns with less than 3 supporting transactions. Step 2.

	A	C	D	E	row sum
1	1	1	1	1	4
3	0	1	1	1	3
7	1	1	1	1	4
8	1	1	1	1	4
10	1	1	1	1	4
11	0	1	1	0	2
col sum	4	6	6	5	

Table 4: Data after eliminating rows with less than 3 items. Step 3.

	A	C	D	E	row sum
1	1	1	1	1	4
3	0	1	1	1	3
7	1	1	1	1	4
8	1	1	1	1	4
10	1	1	1	1	4
col sum	4	5	5	5	

transactions that are involved in frequent itemsets with at least n items, where by frequent we mean that the itemset must appear in m or more transactions. Instead of trying to find the interesting frequent patterns, we are asking for something simpler—for the items and transactions involved in such patterns.

We will use the data set provided in Table 1. Transactions have the numeric labels 1 through 12, while items are labeled with the letters ‘A’ through ‘E.’ This table also shows the sums of each row and column, which represent, respectively, the number of items in each transaction and the number of transactions containing each item. For example, transaction 1 contains 4 items, while item A occurs in 6 transactions.

2.1 The Support Envelope

We start with a simple idea: Any itemset that has a support requirement of m must contain items that occur in at least m transactions. Similarly, any itemset that contains n items must be supported by transactions that have at least

n items. Thus, for example, suppose we are interested in all itemsets that contain at least $n = 3$ items and have at least $m = 3$ supporting transactions. From Table 1, we see that we need consider only transactions (rows) 1, 3, 7, 8, 10, and 11, since all other transactions have fewer than 3 items. We are not saying that these transactions are definitely involved in frequent itemsets with 3 or more items, only that they might be.

Turning our attention to items, we now evaluate which items are involved in itemsets of size 3 or more with a support of 3 or more. We might first look at the number of transactions in which each item appears—see Table 1—but such an approach does not take into account our previous observation that some transactions should be eliminated. In particular, it is only the support of items within transactions 1, 3, 7, 8, 10, and 11 that is important, as these are the only transactions capable of supporting an itemset with 3 or more items. After computing support within only these transactions, we get the support for the five items to be, in order, 4, 2, 6, 6, 5—see Table 2. Thus, we can eliminate item B.

Of course, once we have eliminated item B, we have invalidated our previous counts of how many items occur in each transaction. After taking the elimination of item B into consideration, the counts for transactions 1, 3, 7, 8, 10, and 11 are, respectively, 4, 3, 4, 4, 4, and 2—see Table 3. Thus, transaction 11 can be eliminated—see Table 4. Further iterations yield no change, and thus, we obtain the result that all itemsets with 3 or more items and 3 or more supporting transactions occur among some subset of items A, C, D, and E and some subset of transactions 1, 3, 7, 8, and 10. This set of items and set of transactions is a support envelope.

To verify the correctness of this result, we computed the frequent itemsets with a support threshold of 3. These itemsets are shown in Table 5 and are consistent with the results of our example, i.e., they show that the only itemsets with support of 3 or more are those involving items A, C, D, and E. It is straightforward to verify—see tables 1 and 4—that the only transactions that contain at least three of these items are 1, 3, 7, 8, and 10.

Table 5: Frequent itemsets with support 3 or more.

1 item	2 items	3 items	4 items
A B C	AC AD AE BD	ACD ACE	ACDE
D E	CD CE DE	ADE CDE	

2.2 The Support Lattice and Boundary

For the data set in Table 1, all the support envelopes can readily be computed by using the algorithm that we developed above for each (m, n) pair, where $1 \leq m \leq 12$ and $1 \leq n \leq 6$. (This algorithm, which we call the Support Envelope Algorithm (SEA), is formally described in Section 3.3.) Not all values of m and n will yield non-empty support envelopes, and some (m, n) pairs may yield the same support envelope, i.e., the same set of transactions and items. Support envelopes can be organized as a lattice [4] by defining one support envelope to be ‘less than’ (a subset of) a second support envelope if the items and transactions of the first are a subset of the items and transactions of the second.

In Figure 1, we see a tree-like display of the support lattice. Nodes represent individual support envelopes, while lines represent the subset relationships between envelopes, i.e., that a lower envelope is a subset of a higher envelope.

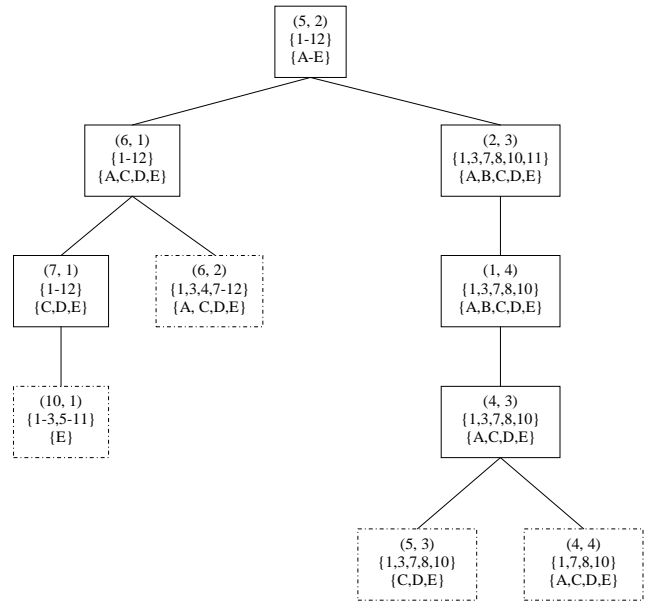


Figure 1: Support lattice for sample data.

Inside each box representing a support envelope are the values of m and n that characterize the envelope, as well as the sets of transactions and items involved in the envelope. Thus, for the envelope represented by the topmost node in Figure 1, each of the items in the support envelope occurs in at least 5 transactions and each transaction in the support envelope contains at least 2 items. Since these conditions are satisfied by all transactions and items, this *root* (or *top*) envelope contains all the transactions and items in the data set. While not illustrated by Figure 1, support envelopes can have multiple parents and/or children.

Since each pair of integers (m, n) is associated with a support envelope (possibly the empty support envelope), we can represent the entire set of support envelopes as a plot of an M by N grid (M is the number of transactions and N is the number of items), where each grid cell is shaded according to its associated support envelope, and where each support envelope is assigned a different shade of gray. Figure 2 shows the set of support envelopes using such a representation. Once again, we have used the same shade of gray for all grid cells, i.e., (m, n) pairs, that represent the same support envelope. Thus, the block in the upper left corner of Figure 2 represents the fact that all support envelopes with $m \leq 5$ and $n \leq 2$, are identical, i.e., contain the same transactions and items. To identify the support envelope associated with each block in the figure, we find the largest possible values of m and n of the block, i.e., we use the values of m and n from the lower right corner of the block. Thus, for example, the big dark block in the upper left corner is the $(5, 2)$ support envelope. Also, as we will discuss in the next section, if a support envelope is below (to the right of) another support envelope, then it is a subset of that envelope, i.e., the transactions and items of the lower (more rightmost) support envelope are subsets of the transactions and items of the higher (more leftmost) support envelope.

For larger data sets, it is more convenient to view the set of support envelopes as a scatter plot of the (m, n) values. Such a plot is shown in Figure 3. To provide more informa-

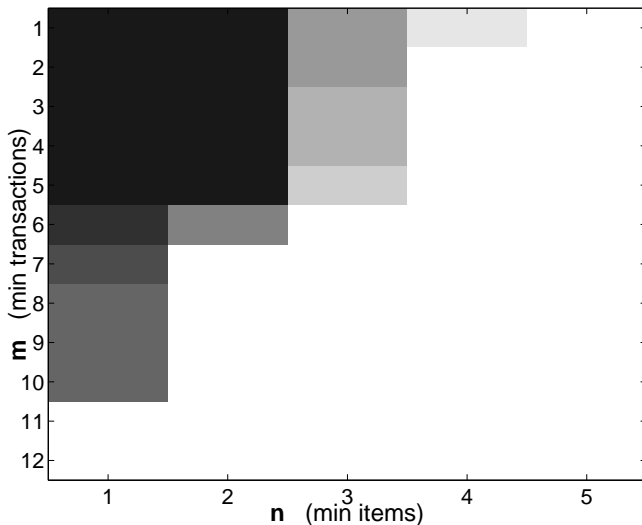


Figure 2: Support lattice for sample data - rectangular format.

tion, we have shaded each point representing a support envelope according to its density, i.e., the fraction of ones in the block defined by the transactions and items in the envelope. (This information can be gathered at almost no cost when finding the support envelopes.) When a support envelope is relatively dense, e.g., more than 50% dense, the envelope is often an interesting pattern in own right, not just because of the information it provides about other association patterns. (All the envelopes in our example are relatively dense.) We show examples of such interesting dense envelopes for real data sets later in this paper.

A scatter plot does not explicitly show all the information of the lattice, i.e., which envelopes are the immediate descendants or parents of other envelopes. However, most of this information is captured by the position of the points that represent envelopes. Specifically, an envelope (point) that is below and/or to the right of a second envelope (point) is contained by that envelope. In our graphical exploration of support envelopes later in the paper, we will focus on the scatter plot of all the envelopes of a data set.

3. SUPPORT ENVELOPES: A FORMAL INTRODUCTION

After a quick review of notation and some other preliminaries, we define support envelopes, analyze the algorithm for finding support envelopes, and discuss some properties of support envelopes.

3.1 Preliminaries

In this section we take care of a few preliminaries.

3.1.1 Notation

An overview of notation is provided in Table 6.

3.1.2 Patterns Characterized by (m, n)

Support envelopes capture the structure of association patterns that can be characterized with respect to a minimum level of support (m), and a minimum number of items (n). We make this more formal in the following definition.

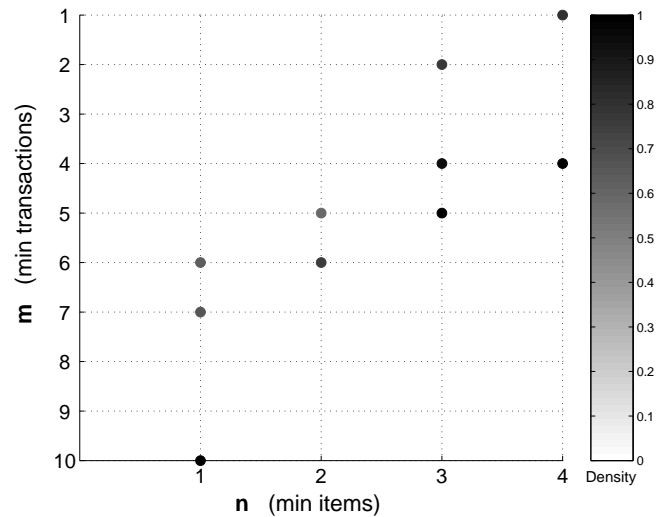


Figure 3: Support lattice for sample data - scatter plot.

Table 6: Summary of Notation

Notation	Description
\mathcal{D}	Data matrix of M rows and N columns
$\mathcal{T} = \{t_1, t_2, \dots, t_M\}$	Set of transactions (rows) of \mathcal{D}
$\mathcal{I} = \{i_1, i_2, \dots, i_N\}$	Set of items (columns) of \mathcal{D}
X, Y	A set of items
R, S	A set of transactions
m	Minimum number of transactions for an envelope
n	Minimum number of items for an envelope
mm	Total number of transactions in an envelope
nn	Total number of items in an envelope

DEFINITION 1. *Pattern Characterized by (m, n)*

A pattern characterized by a pair of positive integers (m, n) is a pattern that involves a set of items X and a set of transactions R such that (a) every item of X occurs in at least m transactions of R and every transaction of R contains at least n items of X , and (b) m and n are maximal.

Many common types of association patterns fall into this category, e.g., frequent itemsets and variations such as closed and maximal itemsets [8]. Error-tolerant itemsets (ETIs) [19], or more properly a slight variation of ETIs, can also be considered to be a pattern characterized by a pair of integers (m, n) , as we show in the following discussion

3.1.3 Symmetric Error-Tolerant Itemsets

An ETI is a frequent itemset that allows a specified fraction ϵ of the items to be missing from any one transaction, e.g., if $\epsilon = 0.2$, then 1/5 of the items in a transaction can be missing. While ETIs are an important approach for extending association patterns, there is a problem with ETIs as currently defined. For example, consider an ETI with $\epsilon = 0.2$, which consists of a set of 10 transactions and 5 items. While each transaction must contain at least 4 out of

5 items, items are not necessarily treated ‘equally,’ e.g., each transaction could be missing the same item. The originators of ETIs noted this issue, which can be problematic in the degenerate case just mentioned, and avoided this degenerate case in an unspecified manner in their implementation. However, it seems desirable, even beyond the current discussion, to modify the notion of an ETI to be more in line with that of frequent itemsets, where all items play an equal role in the pattern. To this end, we define a symmetric ETI.

DEFINITION 2. Symmetric Error Tolerant Itemset

A symmetric ETI is a frequent itemset that allows a specified fraction ϵ of the items to be missing from any one transaction and a specified fraction β of transactions to be missing a particular item.

Any symmetric ETI can be characterized by a pair of integers (m, n) . For example, assume that we have a symmetric ETI which has 12 transactions and 4 items, where $\epsilon = 1/4$ and $\beta = 1/3$. Then, $m = 8$, i.e., each item must appear in 8 transactions in the ETI, and $n = 3$, i.e., each transaction must contain at least 3 of the items in the ETI. Thus, symmetric ETI’s are contained in support envelopes. The importance of this is that the support boundary is not only a bound on frequent, closed, and maximal itemsets, but also on patterns, such as symmetric ETIs, that are more general.

On a theoretical note, a support envelope can be considered as a type of symmetric ETI. Suppose that we have a support envelope characterized by (m, n) . If mm is the total number of transactions in the envelope and nn is the total number of items in the envelope, then the support envelope is a symmetric ETI with $\epsilon = n/nn$ and $\beta = m/mm$. However, support envelopes have properties not shared by all symmetric ETIs.

3.2 Definition of a Support Envelope

We now formally define a support envelope.

DEFINITION 3. Support Envelope

A support envelope $E_{m,n}$ characterized by a pair of positive integers (m, n) is a set of items $X \subseteq \mathcal{I}$ and a set of transactions $R \subseteq \mathcal{T}$ which contains all patterns characterized by any pair of positive integers (m', n') such that $m' \geq m$ and $n' \geq n$.

Expressed another way, the transactions and items of the support envelope are the union, respectively, of all the transactions and items of all patterns characterized by any (m', n') , such that $m' \geq m$ and $n' \geq n$. Also, by construction, there cannot be more than one support envelope per (m, n) pair. However, as discussed previously and below, the support envelopes for different pairs of integers (m, n) may be identical, and in that case, we will refer to the support envelope using the largest values of m and n .

3.3 The Support Envelope Algorithm (SEA)

The algorithm for finding support envelopes—SEA—is shown below. We analyze the complexity, convergence, and correctness of SEA, as well as discussing how SEA can be used as the basis of algorithms for finding either the envelopes in the support boundary or the support lattice. Initial implementations of these algorithms are available from [16].

Algorithm 1 Support Envelope Algorithm (SEA).

- 1: {Input: Data matrix \mathcal{D} and a pair of positive integers m and n }
- 2: **repeat**
- 3: Eliminate all rows whose sum is less than n .
 (Eliminate all transactions with fewer than n items.)
- 4: Eliminate all columns whose sum is less than m .
 (Eliminate all items in fewer than m transactions.)
- 5: **until** \mathcal{D} does not change
- 6: **return** the set of rows (transactions) and columns (items) remaining in \mathcal{D}

Table 7: Evaluation Data Sets

Data Set	# Transactions	# Items	Density
chess	3196	75	0.49
mushroom	8124	119	0.19
LA1	3204	31472	0.0048
kosarak	990,002	41,270	0.00019

Table 8: Statistics of Computing Support Envelopes

Data Set	# Envelopes	Time (sec)	Avg Iter
chess	353 (37)	3.4 (0.6)	2.1
mushroom	535 (22)	4.7 (0.8)	2.1
LA1	26,678 (168)	570.2 (29.6)	3.1
kosarak	110,606 (318)	46,807 (390.8)	3.0

Some data sets that will be used to illustrate various aspects of the following algorithms are listed in Table 7. The first three data sets can be found at the Frequent Itemset Mining Implementations Repository [6], while the LA1 data set is from the LA data of TREC-5 [22]. Table 8 shows various statistics related to the computation of support envelopes for the given data sets. (The results in this table were gathered by experiments performed on a 3.2 GHz Intel Xeon® Linux system.) The first value in the # Envelopes column is the number of envelopes for a data set, while the value in parentheses is the number of envelopes on the support boundary. The first value in the Time column is the time (in seconds) required to compute all the support envelopes for a data set, while the value in parentheses is the time (in seconds) to compute the support boundary.

Complexity: The time complexity of SEA is proportional to (a) the number of iterations, num_iter , and (b) the amount of time to test each row and column to see if it should be eliminated. For a sparse matrix representation, the time to sum a row or column and compare this sum to m or n , is proportional to the number of non-zero entries in the row or column. Therefore, the total time required to check which rows and columns should be eliminated is proportional to, nnz , the number of non-zero entries in the entire matrix, and the overall time complexity of SEA is $O(M * N * nnz * num_iter)$.

If a parent of a support envelope is known, then finding a support envelope is somewhat more efficient because we can use the transactions and items of the parent envelope to provide a smaller starting matrix. This effect is more important for larger data sets. For example, with the kosarak data set shown in Table 7, the average number of iterations is reduced from 8 to 3 by using a parent envelope. For the LA1 data set, which is significantly smaller, the average number of iterations is reduced from 4 to 3. For mushroom and chess, the reduction is from 3 to 2.

The space required for SEA is $O(nnz)$ plus the amount

of space required to store the support envelopes. While the row and column indices can potentially require significant space, there are a couple strategies that can be used to significantly reduce memory requirements. First, we can store only the indices in which an envelope differs from a parent envelope. (Preliminary investigations indicate that this approach seems promising.) A second approach is to not store the row and column indices at all since the entire envelope can be quickly recomputed if needed.

Convergence: We show that SEA converges to a solution in a fixed number of steps.

THEOREM 3.1. *SEA converges to a solution in a fixed number of steps whenever the input is a binary matrix \mathcal{D} and positive integers m and n .*

PROOF. Either SEA eliminates a row or column at each step, eventually producing an empty matrix, or else SEA terminates after a step in which \mathcal{D} is unchanged. Thus, SEA will converge to a solution in at most $\min(M, N)$ steps. \square

As mentioned, typically, convergence occurs in a much smaller number of steps, i.e., less than 10 for the data sets in Table 7.

Correctness: Here we show that SEA finds support envelopes.

THEOREM 3.2. *Given the data matrix \mathcal{D} and a pair of positive integers (m, n) , the set of items $Y \subseteq \mathcal{I}$ and transactions $S \subseteq \mathcal{T}$ returned by SEA are the support envelope of \mathcal{D} characterized by (m, n) , i.e., $SEA(\mathcal{D}, m, n) = E_{m,n}$.*

PROOF. Consider any pattern that is characterized by $m' \geq m$ and $n' \geq n$ and that involves the set of items $X \subseteq \mathcal{I}$ and the set of transactions $R \subseteq \mathcal{T}$. By Definition 1, every transaction in R must contain at least n items of X and every item in Y must occur in at least m transactions of R . Consequently, none of the transactions of R can be eliminated in Step 3 of SEA since they are ‘supported’ by the items of Y . Also, none of the items of Y can be eliminated in Step 4 of SEA since they are ‘supported’ by the transactions of R . Thus, $X \subseteq Y$ and $R \subseteq S$, i.e., the items and transactions of this pattern are subsets of the set of items and transactions found by SEA. From this, we conclude that the items and transactions returned by SEA contain those of $E_{m,n}$. We still need to show that $E_{m,n}$ contains the items and transactions returned by SEA. However, the set of items and transactions returned by SEA is certainly a pattern characterized by (m, n) and must belong to $E_{m,n}$. \square

An Algorithm for Finding the Support Boundary

A straightforward approach to finding the support boundary is to perform a binary search over possible values of n for each value of m , or to perform a binary search over possible values of m for each value of n . Such an approach results in a conservatively estimated time complexity of $O(\min(N \log(M), M \log(N)) * nnz * num.iter)$. (Practically, this means that it is somewhat better to perform the binary search over values of n if there are fewer rows than columns, and vice-versa.) We omit a detailed description of this algorithm. An implementation of this algorithm can be found at [16] and the statistics from runs for real data sets are given in Table 8.

Algorithms for Finding All Support Envelopes A simple approach to finding all the support envelopes is, for each value of m , to find envelopes for each value of n up

to the limit imposed by the support boundary. (Again, this means that it is somewhat better to perform our algorithm by varying n for a fixed m if there are fewer rows than columns, and vice-versa.) An implementation of this algorithm can be found at [16], and statistics from runs for real data sets are presented in Table 8.

Scalability A basic strategy for parallelizing the previous algorithm is to have each processor compute all the envelopes for a fixed value of m (or n) and all values of n (or m). A similar approach could be used to parallelize the algorithm for finding the support boundary. Issues that would need to be considered include load balancing and the generation of duplicate support envelopes by different processors.

Another important scalability issue that needs investigation is how to best adapt the SEA algorithm for cases where the data is large, i.e., where the data does not fit in main memory and it is desirable to (a) access the data in a sequential manner and (b) make as few passes over the data as possible. SEA can be implemented to access data in a sequential manner if two copies of the data are kept, one with the data ordered by rows and the other with the data ordered by columns.

3.4 Properties of Support Envelopes

We have already encountered a number of the properties of support envelopes in the previous discussion. For completeness and clarity, we summarize the properties of support envelopes in this section. Actual proofs of these properties can be found in our technical report [17].

For for each pair of integers (m, n) , where $1 \leq m \leq M$ and $1 \leq n \leq N$, there is an associated support envelope—perhaps the empty support envelope—which is the support envelope that SEA will return when given (m, n) and the data set \mathcal{D} . However, as we saw in the example, the same support envelope may be associated with more than one pair of integers since the items and transactions that satisfy the (m, n) constraints may actually satisfy stronger constraints. We characterize a support envelope by the strongest support and item constraints (i.e., by the maximum (m, n) values) that it satisfies. A consequence of these two facts is that the number of support envelopes in a data set is at most $M * N$.

One support envelope may contain another support envelope, i.e., the items and transactions of one envelope may be subsets of the items and transactions of another envelope. Said another way, a support envelope defines a block of the data set (matrix) \mathcal{D} , and this block may contain the blocks that correspond to other envelopes. Indeed, for a given support envelope characterized by (m, n) , any envelope characterized by (m', n') , such that $m' \geq m$ and $n' \geq n$, must be a subset of the (m, n) envelope. This is not surprising, since increasing m and n will likely eliminate items and transactions.

Like frequent itemsets, support envelopes form a lattice. For support envelopes the order relationship is the subset relationship described above. Support envelopes that contain no other (non-empty) envelopes are said to be on the support boundary.

An interesting property that was not mentioned in the example, is that the density of envelopes, i.e., the fraction of 1’s in the block represented by the envelope, increases as we move down the lattice. Thus, if one envelope contains another, it must be less dense (or at least no more dense) than the envelope that it contains. This means that

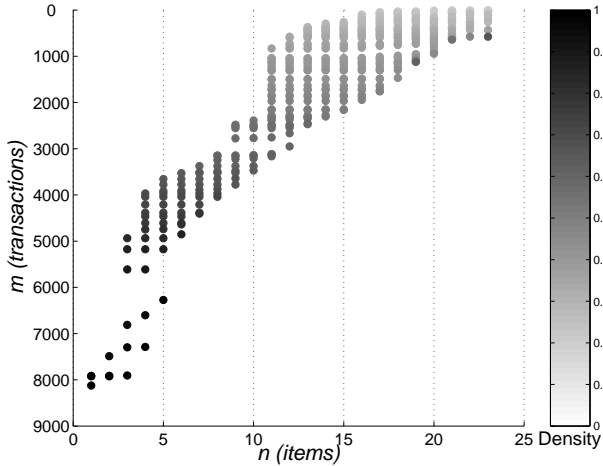


Figure 4: Support envelopes for mushroom data set.

the support boundary must contain the densest envelopes in the support lattice. This does not mean, however, that every envelope on the support boundary is denser than any envelope elsewhere in the support lattice.

There can be at most $\min(M, N)$ envelopes on the support boundary. Otherwise, two envelopes on the boundary would have the same value of m or n and one would contain the other. For a similar reason, the values of m and n are inversely related as we move along the support boundary, i.e., if we order the support envelopes to have increasing values of m , then successive envelopes will have decreasing values of n .

4. EXPLORING TRANSACTION DATA SETS WITH SUPPORT ENVELOPES

In this section we demonstrate how support envelopes can be used to explore the transaction data sets of Table 7.

4.1 Dense Data Sets: Mushroom and Chess

We begin by showing the support envelopes for the mushroom and chess data sets in figures 4 and 5, respectively. As before, we have shaded the points to show the density of the support envelopes, i.e., the fraction of 1’s in the block of items and transactions defined by the envelope. Both of these data sets are relatively dense compared to many transaction data sets and thus, the overall densities of the envelopes are relatively high. For both data sets, the densities of the envelopes are highest for envelopes with many transactions, but few items. This reflects the fact that both data sets have a few items that occur in most transactions. In some cases, these ‘overly frequent’ items should be discarded since they yield little information.

Also, for both data sets, there are no envelopes near the origin. This reflects the fact that every item occurs in some minimum number of transactions and that every transaction contains some minimum number of items. For mushroom, every item occurs in at least 4 transactions and every transaction contains exactly 23 items. For chess, every transaction contains 37 items and every item occurs in at least 1 transaction.

However, all association patterns characterized by (m, n) values for which there are no points in the figures, can be found in the transactions and items of the envelopes on the

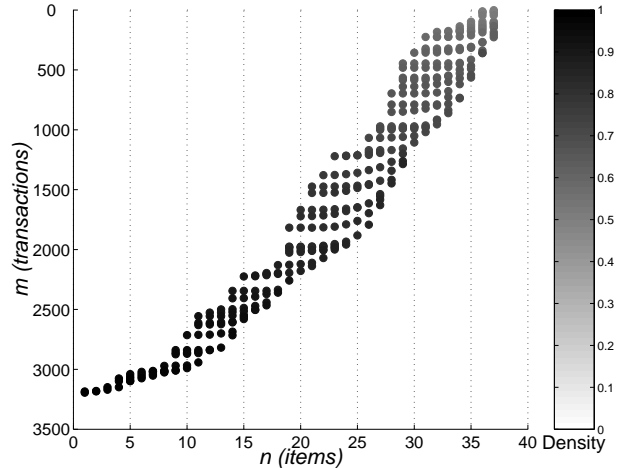


Figure 5: Support envelopes for chess data set.

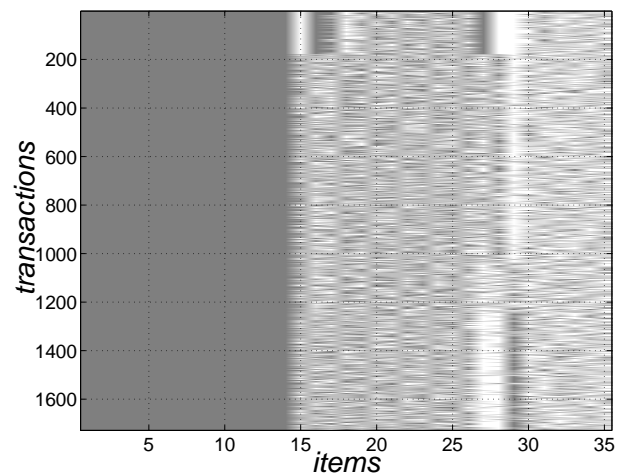


Figure 6: Plot of mushroom support envelope (576,23).

inner boundary of the support lattice. Indeed, the root envelope corresponding to $(1, 1)$ —and perhaps other pairs of integers—is guaranteed to encompass all association patterns in the data set. For mushroom, the root envelope is characterized by the pair $(4, 23)$, while for chess, the root envelope is characterized by $(1, 37)$.

Despite some similarities, the two sets of support envelopes are quite distinct, e.g., the boundary of the mushroom data set curves inward (towards the origin), while that of the chess data set curves outward. Our observation, which is based on a limited number of data sets, is that for sparser data, the boundary is basically concave, while for denser data sets, the boundary is basically convex. Of course, as figures 4 and 5 show, regardless of the overall curvature of the boundary, it may—in different sections—actually be both concave and convex.

As another example of how we might use support envelopes to analyze the structure of support patterns in a data set, we remark that there is an unusually dense support envelope with 576 transactions and 23 items in the mushroom data set. It has a density of 0.65, while most of the envelopes around it have densities ranging from 0.3 to

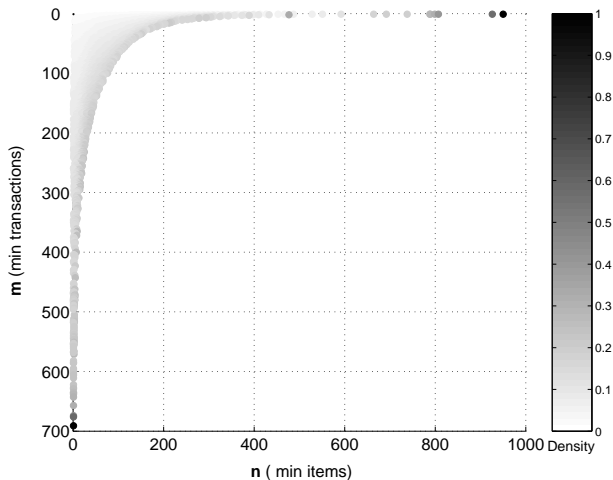


Figure 7: Support envelopes for LA1 data set.

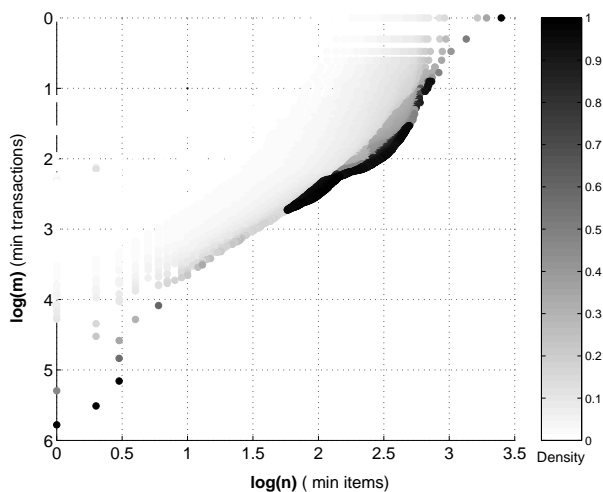


Figure 8: Support envelopes for kosarak data set. Note: log scale.

0.4. Since this support envelope is somewhat anomalous, we show its plot in Figure 6. The items and transactions of the envelope have been sorted to put the denser region of the support envelope toward the upper left hand corner. Each gray pixel corresponds to a 1 in the envelope, while each white pixel corresponds to a 0. The figure shows that there is a group of 14 items that occur together in every one of the 1728 transactions in the envelope. (Remember that the total number of transactions (mm) and items (nn) in an envelope is different from the m (support) and n (item) constraints.)

Upon further analysis, we discovered that one of the columns was the column 48, ‘gill-color:buff.’ There are exactly 1728 instances of item 48, every one of which occurs with 13 other items (one of which is ‘poisonous’). This support envelope is somewhat denser than the others around it because the co-occurrence of 13 items is larger than is typical for this data set.

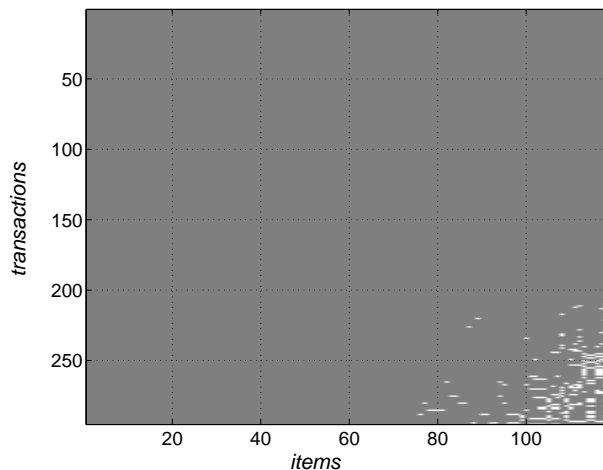


Figure 9: Plot of kosarak support envelope (276,113).

4.2 Sparse Data Sets: LA1 and kosarak

For contrast, we show a graph of the LA1 support envelopes in Figure 7. LA1 is much more sparse and has more items, although fewer transactions, than either mushroom or chess. The support envelopes are relatively sparse compared to those of both mushroom and chess. However, the density does increase from 0.0048 for envelopes near the origin (upper left) to around 0.2 for envelopes at the boundary. The two black dots in the upper right and lower left represent, respectively, the longest document and the most frequent item (word). The support boundary is extremely concave in LA1.

The set of kosarak envelopes is shown in Figure 8. We have used \log_{10} scales for both the m and n axes because the m and n values have a much wider range of values. However, if this were not the case, the support boundary of kosarak would be even more concave than that of LA1.

Interestingly, some of envelopes on or near the kosarak support boundary, with values of m between 100 and 1000 (between values of 2 and 3 on the \log_{10} scale) have relatively high density. We investigated the boundary support envelope (276, 113). Note that it would be difficult to investigate patterns with such low support using traditional approaches. Indeed, the results of the FIMI workshop [6], show that current algorithms to find frequent, closed, or maximal itemsets start to experience a sharp growth in run time at a support threshold of about 0.1% (990) transactions.) The support value for envelope (276, 113) is even lower than this, i.e., less than 0.03%.

Figure 9 shows a plot of the support envelope corresponding to (276, 113), which has a density 0.993. This envelope contains a frequent itemset of size 250, and we tried to find this itemset using the algorithm, `fpmmax`, which is available from the FIMI website [6]. The program ran for more than a week on a 2.8 GHz Intel Xeon[®] and produced a file with almost 5 million maximal itemsets. (We are not sure whether the program terminated normally since memory usage had grown to 1.2 GB shortly before it terminated.) In contrast, the support boundary for kosarak was computed in approximately 15 minutes on the same machine. (Computing the boundary on a faster machine took only about 7 minutes.)

5. EXTENSIONS OF THE BASIC APPROACH

In this section, we discuss two additional extensions of the basic approach: computing support envelopes with respect to specific transactions or items and adding additional constraints to support envelopes.

5.1 Transaction Specific Support Envelopes

When support envelopes are not very dense, they are of interest only for the patterns that can be extracted from them and for what they tell us about the overall structure of association patterns. However, if a support envelope is relatively dense, then the envelope is interesting as a pattern in its own right. Thus, it may be useful to seek special situations where support envelopes are more likely to be dense. To that end, we consider support envelopes associated with a particular transaction (or item).

To illustrate this idea, we show an example using the LA1 data set. We constructed a data set specific to the first document (transaction) in LA1 by eliminating all words (items) that do not appear in this document. (This document comes from the ‘Financial’ class.) We then computed the support boundary of this document specific data set. Since we are working in the document domain, we display the results in a table that shows the parameters of the support envelopes and the words (items) that are part of the support envelopes. (LA1 was processed using standard information retrieval techniques, e.g., the words are stemmed and stop words are eliminated.) The first support envelope consists of the selected document and all its words, while successive envelopes consist of smaller subsets of these words. For successive envelopes along the boundary, the number of documents increases, while the number of words (usually) decreases. In other words, the pattern represented by the support envelope is becoming more general as we move along the support boundary from lower to higher values of m . The most interesting support envelopes are those in the middle, since they are not as specific as a particular document, nor so general as to be uninteresting. The envelope with the highest value of m contains the word, ‘home,’ which is the most frequent word contained by the first document.

These support envelopes have a clear theme of a bank bailout of the home savings and loan industry. However, the patterns represented by these support envelopes are only moderately strong, i.e., the support envelopes that contain more than a few documents and items tend to have densities in the range of 0.5 to 0.7. Nonetheless, recall that the regular support envelopes for LA1 only have a density that is, at best, around 0.2.

5.2 Constrained Support Envelopes

A special case of extending support envelopes that is easy to implement and interpret involves adding a constraint to the support envelope process. For example, very frequent items show up in many association patterns, but provide little useful information. One way to address this issue is to require that the fraction of an item involved in an association pattern meets some minimum threshold. (This is formally defined by the notion of a hyperclique pattern [18].) It is straightforward to add such a constraint to the computation of support envelopes. In other words, given a specified fraction f between 0 and 1, we eliminate an item

Table 9: Support boundary envelopes LA1 restricted to words (items) in the first document (transaction).

m	n	mm	nn	words
1	26	1	26	bailout bank billion board clos cost countri deal expect feder goal govern hemorrhag home hoyle industri karl loan overall quote rest sav spokesman stop texa throughout
2	16	2	16	bailout bank billion board clos cost deal expect feder govern home industri loan rest sav texa
3	13	3	13	bailout bank billion board cost deal feder govern home industri loan sav texa
5	11	10	15	bailout bank billion board cost countri deal expect feder govern home industri loan rest sav
7	10	30	21	bailout bank billion board clos cost countri deal expect feder goal govern home industri loan rest sav spokesman stop texa throughout
13	9	38	17	bank billion board clos cost countri deal expect feder govern home industri loan sav spokesman stop throughout
21	8	46	14	bank billion board clos cost countri deal expect feder govern home industri loan sav
33	7	80	14	bank billion board clos cost countri deal expect feder govern home industri loan sav
54	6	150	14	bank billion board clos cost countri deal expect feder govern home industri loan sav
80	5	197	11	bank billion board cost countri deal expect feder govern home industri
121	4	348	11	bank billion board cost countri deal expect feder govern home industri
193	3	552	9	board cost countri deal expect feder govern home industri
298	2	532	4	countri expect govern home
676	1	676	1	home

Table 10: Support boundary envelopes LA1 restricted to words in the first document and with the additional constraint that 50% of the occurrences of a word must be in the envelope.

m	n	mm	nn	words
140	3	894	16	bank billion board clos cost countri deal expect feder goal govern home industri rest spokesman stop
287	2	669	5	countri expect feder govern home
676	1	676	1	home

if does not occur in at least m transactions and at least f of its total support occurs among the transactions of the support envelope. For example, if $f = 0.5$, then at least half of the supporting transactions of an item must occur in the support envelope. While a similar constraint could also be applied to the rows as well, we do not pursue that approach here.

To illustrate this approach, we find the support boundary specific to the first document in the LA1 data set using a threshold of 0.5. The results are shown in Table 10. There are far fewer envelopes, but they still seem to capture much of the meaning of the envelopes of Table 9. In particular, the first envelope on the boundary no longer consists of all the terms from the first document of LA1 that was used to generate the reduced data set.

6. RELATED WORK

To our knowledge, the notion of a support envelope is new, although the concept of an ‘envelope’ is common in mathematics, where an envelope is a mathematical entity—usually a curve—that bounds a collection of other mathematical entities. The related notion of a cover has been used before in association analysis, but for association rules [3, 12, 14].

The idea of support envelopes was inspired partly by various concepts in lattice theory [4] and formal concept analysis [5], especially those ideas that have found their way into association analysis [5, 7, 11]. In particular, a key motivating concept for us is the notion of efficiently representing frequent itemsets via a lattice of closed itemsets [1, 13, 21]. More generally, closed itemsets are a specific example of con-

densed representations [2, 10]. However, support envelopes are not a condensed representation since it is necessary to keep the original matrix if the goal is to use support envelopes to actually find itemsets.

The idea of error tolerant itemsets (ETIs) [19] also played an important role in our thinking. In particular, ETIs emphasize the notion that it is useful to consider a version of frequent itemsets that relaxes the requirement that all items be contained in all transactions. Support envelopes also embrace this idea, and as we showed in Section 3.1.3, are a special type of ETI.

Finally, there has been some recent work on computing tight lower bounds for distributions of frequent and maximal frequent itemsets [15]. Although this is a different line of inquiry than we have pursued in this paper, such work is also quite relevant to understanding the structure of support based patterns in transaction data.

7. CONCLUSIONS AND FUTURE WORK

In this paper, we introduced support envelopes, a new tool that is useful for exploring the high-level structure of association patterns in a transaction data set. Support envelopes are not encumbered by a support threshold and they provide information not only about frequent, closed, and maximal itemsets, but also about more general patterns, such as symmetric error-tolerant itemsets. Support envelopes provide both a theoretical basis for understanding the structure of association patterns and a graphical technique for visualizing this structure. Furthermore, there are simple and efficient algorithms to compute a single support envelope, the support envelopes on the support boundary, or all the support envelopes of a data set.

There is considerable potential for future work. Current implementations for finding the support boundary support or all support envelopes and could be improved and parallelized to provide additional performance. Also, the theoretical properties of support envelopes should be further explored. In particular, the extensions of support envelopes deserve further investigations since they may be useful both for finding actual patterns, as well as for providing additional information about the overall structure of association patterns. In particular, we hope to investigate whether we can extend the notion of support envelopes to continuous data. Finally, further work is necessary to more fully understand what information can be extracted from scatter plots of support envelopes.

8. ACKNOWLEDGMENTS

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