Bearing-Only Pursuit

Nikhil Karnad and Volkan Isler

***Abstract*— We study a variant of a well-known pursuit evasion game, the lion and man game. In this game a lion (the pursuer) tries to capture a man (the evader). The players move in turns. At each time step, they can move a unit distance. We focus on a version which takes place in an unbounded arena: the positive quadrant of the plane. The novelty of our formulation is in the sensor model. In the original formulation, the lion can sense the precise location of the man at all times. In our version, which is inspired by mobile robots equipped with monocular vision systems, the lion can only obtain bearing information about the man's location. We present a pursuit strategy which guarantees that the distance between the players is reduced to the step size in a bounded number of steps.**

I. INTRODUCTION

Pursuit-evasion games are problems of fundamental interest in Robotics. In a pursuit-evasion game, one or more pursuers try to capture an evader while the evader tries to avoid capture. Many problems arising from diverse applications such as collision-avoidance, search-and-rescue, air-traffic control and surveillance can be modeled as pursuit-evasion games.

In most pursuit-evasion game formulations, the players can observe each other's state (e.g. position) at all times. However, in some robotics applications, the pursuer may not have the necessary sensors to obtain the evader's position. In this paper, we focus on such a scenario and study a variant of a fundamental pursuit-evasion game, the lion-and-man game [1], [2], [3]. In the original version of this game, a lion (pursuer) tries to capture a man (evader) inside a circular arena by moving onto the man's current location. The players are both holonomic. They have the same speed and can observe each other's locations.

In our version, we restrict the pursuer's observation capabilities. The pursuer can not directly observe the evader's location. Instead, it has access to the readings of a bearing-only sensor such as a camera. In other words, the pursuer can obtain a ray that contains the evader but can not measure the location of the evader on this ray. The main question we seek to answer is whether the pursuer can capture the evader under this limitation.

Before we formalize the game model, we start with an overview of the related work.

A. Related work

There are numerous versions of pursuit-evasion games. In this section, we focus on only the lion-and-man game and present related work.

The original lion-and-man game takes place in continuous time and is played inside a circular arena. The first solution to this game, proposed by Besicovitch, is as follows: the lion moves to the center of the disk. Afterward, it remains on the radius that passes through the man's position. Since the players have the same speed, the lion can remain on the radius and simultaneously move toward the man. It turns out that the temporal aspect of the game is crucial in determining the outcome. First, consider the discrete-time version, where at every time step, the players move in turns. For this version, Besicovitch's solution clearly guarantees capture. Surprisingly, Littlewood showed that when the game takes place in continuous time, the man wins! While the lion can get arbitrarily close to the man in finite time, capturing the man takes forever. See [1] for an overview of these results. For the continuous-time version, Alonso et al. presented an almost-optimal strategy by showing that the lion can get within a distance c of the man in time $O(\frac{r}{s}\log\frac{r}{c})$ where r is the radius of the arena and s is the maximum speed of the players [2].

Sgall studied a variant of the lion-and-man game which takes place in the non-negative quadrant of the plane [3]. This version of the game first appeared in [4]. The outcome of the game depends on the initial positions of the players. Let (x_p, y_p) and (x_e, y_e) be the initial position of the pursuer and the evader respectively. If either $x_e \geq x_p$ or $y_e \geq y_p$, it is easy to see that the evader wins. Sgall showed that in the remaining case, the pursuer wins. He presented an almost optimal strategy that is quadratic in the pursuer's distance from the origin and the slope of the line connecting the player's initial locations.

Recently, Isler et al. showed that the lion can capture the man inside any simply-connected polygon [5]. Alexander et al. presented a sufficient condition for a natural greedy strategy to succeed in arbitrary dimensions [6]. More recently, Bopardikar et al. [7] studied a sensing limitation in the lion-and-man game. In their model, the lion can observe the man's exact location only if the distance between the players is less than a given threshold. In this paper, we focus on a different type of sensing limitation and study pursuit strategies for a pursuer equipped with a bearing-only sensor.

B. Our results

When the game is played inside a circle and in discrete-time, it is easy to see that the lion can capture the man simply by moving toward it along the line l connecting them: to maintain the distance between the players, the man must move away from the lion along l. But since the arena is

V. Isler is with the Faculty of Computer Science, Rensselaer Polytechnic Institute, 110 Eighth Street, Troy, NY, USA isler@cs.rpi.edu

N. Karnad is a graduate student with the Department of Computer Science, Rensselaer Polytechnic Institute, 110 Eighth Street, Troy, NY, USA karnan@cs.rpi.edu

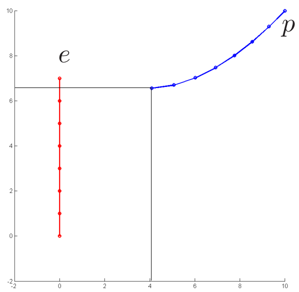

Fig. 1. The greedy pursuit fails. The evader escapes by breaking the invariant that it must remain inside the axis parallel rectangle defined by the pursuer's location.

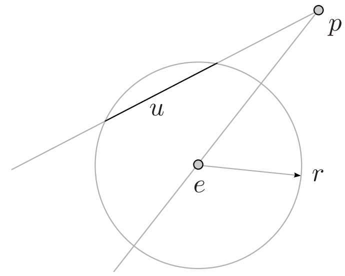

Fig. 2. After an evader move from e, the pursuer can infer that the evader is inside the line segment u by intersecting its bearing measurement (the ray that originates from p and goes through the evader's true location) with the unit ball around e. Here, $r = 1$ is the maximum step size.

bounded, the man must eventually turn – at which point the distance between the players decreases. Though this greedy strategy takes a long time to capture, it requires only bearing information which implies that a pursuer with a bearing-only sensor can capture the man inside a circular arena.

Therefore, in this paper, we focus on the version of the game that takes place in the positive quadrant of the plane as studied in [4], [3]. Note that the greedy strategy described above does not guarantee capture in this version (see Figure 1). Further, the solution proposed in [3] requires the precise location of the evader *before the pursuer move.* Hence, it is not applicable for the bearing-only case. We present a novel strategy for the bearing-only pursuer which guarantees "capture" provided that both of the initial coordinates of the evader is less than the pursuer's corresponding coordinates (otherwise the pursuer can not capture the evader even if he can observe the evader's precise location). An interesting aspect of our strategy is that it requires the pursuer to combine multiple observations and the knowledge about the evader's speed to compute its next move.

In the next section, we start by formalizing the game model.

II. The Game Model

We study a discrete-time, continuous-space game. During a time-step, the players can move at most unit distance. The game is played in the positive first quadrant, with the evader starting at a location between the pursuer and the origin. Initially, we assume that the pursuer is aware of the exact location of the evader (We will remove this assumption later in Section IV-A.). From then on, the pursuer uses bearing-only information to estimate where the evader is. A round of the game consists of the following.

- An evader move in which he moves from his current position to another that is at a distance at most 1 from the previous position.
- The pursuer sensing and estimating where the evader is using the bearing information and the previous evader position (Fig. 2).
- The pursuer making his move to a point at a distance at most 1 away from his current location.
- The pursuer sensing again and intersecting the current ray with the previous estimate to get the exact position of the evader.

Invariant 1. At the end of any round, the evader has to be located between the pursuer and the origin.

Any winning pursuit strategy must maintain this invariant at all times, otherwise the evader escapes by moving parallel to a suitable axis away from the pursuer.

Note that when it is the pursuer's turn to move, he does not know the exact evader position. This limitation prevents the pursuer from employing the lion's strategy described by Sgall [3]. We show that there exists a pursuer strategy which uses a special point on the evader estimate (line-segment u in Fig. 2) to make each of its moves, resulting in overall finite capture time. We call this a *conservative move*, because the pursuer ensures that Invariant 1 is not broken no matter where on the estimate the evader actually lies.

We say that the evader is *captured* and the pursuer wins the game if, at the end of any round the distance between the two players is less than or equal to a constant c. Such a constant exists in practice, because the players are never point objects. In this paper we show that a pursuit strategy which guarantees capture for any $c \geq 1$ exists. Therefore, if the step size is chosen so that the unit distance is smaller than the radius of the evader, our strategy guarantees that the pursuer "hits" the evader. We refer to c as the *capture threshold*.

In the following section, we present the pursuer's strategy and proceed to prove its correctness in Section IV.

III. Pursuer's Strategy

Before explaining the bearing-only pursuit strategy, we give a brief overview of Sgall's Lion strategy.

A. Sgall's Lion strategy

In an earlier paper [3], Sgall presented a pursuer strategy that guarantees capture of the evader in the positive quadrant of the plane, given that the initial conditions satisfy Invariant 1.

Let P and E be the initial coordinates of the pursuer and the evader respectively. Suppose P and E satisfy Invariant 1. Find a point Q on the line EP such that P lies between Q and E, and a circle C centered at Q, passing through P, touches (or cuts) both the X-axis and the Y-axis. The main idea is for the pursuer to make his moves in such a way that the circle C centered at Q and passing through the pursuer's current location advances further and further away from Q until the evader is trapped. The pursuer executes his move in the following manner.

Suppose the evader moved to E' such that $|EE'| \leq 1$ (maximum step size). The ray CE' intersects a circle centered at P of radius 1 at two points. Of these, the pursuer picks the point farthest from Q and moves to it, call it P'. This is the end of a round. We will refer to this move as the *Lion's move with respect to Q*.

Sgall showed that the Lion's strategy ensures capture by proving that (a) the distance of P from the center Q always increases by a lower bound, no matter what the evader does, and, (b) the pursuer is always inside the line segment connecting the evader to center Q. These two conditions guarantee capture of the evader. Note that the pursuer's distance to Q can be viewed as a measure of *progress*. We adopt this terminology in the rest of the paper.

B. Bearing-only strategy

In order to execute Sgall's strategy, the pursuer needs to know the exact location of the evader before making his move. Although our pursuer can use the bearing-only ray to triangulate the position of the evader, he will know the exact evader location only *after* he has moved. Therefore, he cannot use this information to execute an exact Lion's move.

The bearing-only pursuit strategy starts by computing a center Q as in Sgall's strategy. The strategy then proceeds in two main phases: (i) employing the original lion strategy with respect to Q whenever possible and, if not (ii) guarding the pursuer's progress while "catching up" with the evader in a finite number of steps.

We adapt Sgall's Lion strategy by using a conservative estimate of the evader's location: Let the position of the pursuer at time t be $p(t) \in \mathbb{R}^2$ and that of the evader be $e(t) \in \mathbb{R}^2$. After the evader move, the pursuer builds an estimate of the evader position by intersecting his sensing ray with a disc of unit radius centered at $e(t)$. The pursuer assumes that the evader is at some point E^* on this estimate and plays Sgall's Lion strategy. The point E^* is chosen as the point on the estimate farthest from the center of the circle (Q) used in the Lion strategy. By making this move, the pursuer ensures that Invariant 1 is never broken i.e. the evader does not have a guaranteed escape plan even if the guess turns out to be wrong. Therefore, we refer to E^* as the *conservative estimate.*

The following lemma justifies using the conservative estimate.

Lemma 1: Let E^* be the conservative estimate and P^* be the pursuer location after the pursuer's move. If $|P^*E^*| \leq 1$, the evader is captured.

Proof: Suppose the pursuer moved to P^* assuming that the evader was at the conservative estimate E^* and this leads to capture i.e. $|P^*E^*| \leq 1$, as shown in Figure 3. Drop the perpendicular from P^* on to the line PE^*. Since

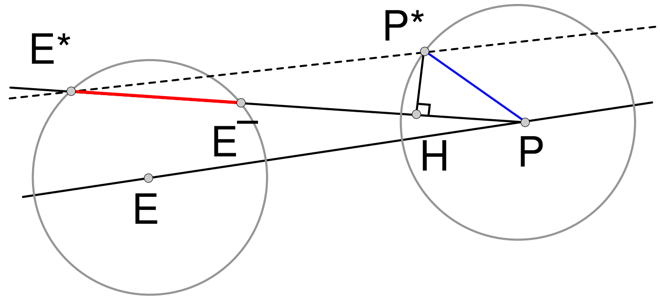

Fig. 3. Capture condition: If the evader is captured w.r.t the conservative estimate i.e. $|P^*E^*| \leq 1$, then he is captured no matter where on the estimate he actually is.

$|PE^-| > 1$ (otherwise the evader would be captured soon after his move), the foot of the perpendicular, call it H, lies inside the circle of radius 1 centered at P. The distance of P^* from the line PE^* is least at H and monotonically increases to $|P^*E^-|$ and then $|P^*E^*|$. Therefore

$$|P^*H| < |P^*E^-| < |P^*E^*| \leq 1$$

which proves that for all points on E^*E^-, the evader is within a distance of 1 from P^*, implying capture. ■

When the pursuer's guess is wrong, the points Q, $p(t+1)$ and $e(t+1)$ are not collinear. The evader is now on one side of the line l through Q and the pursuer. This will trigger the *guarding phase* where the pursuer switches to a *guarding strategy*.

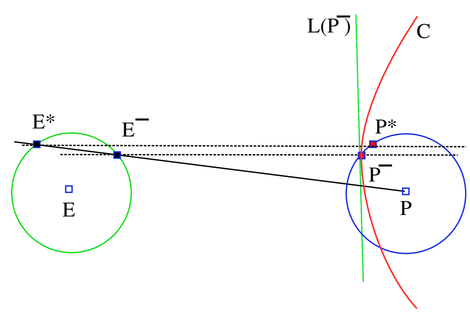

Fig. 4. The guarding phase of the pursuer's strategy, in which he prevents the evader from crossing the line $L(P^-)$.

Call the point he should have been at to continue the Lion's strategy as P^- (Figure 4). Let $L(P^-)$ be the line through P^- tangent to the circle centered at Q passing through P^-. Since the line through E^- and Q is perpendicular to $L(P^-)$, the projection of E^- onto $L(P^-)$ is P^-. If the pursuer were at P^- instead of P^*, he could prevent the evader from crossing $L(P^-)$ just by moving toward the evader's projection on that line. We call this *guarding* the line $L(P^-)$. The guarding strategy involves the pursuer moving from P^* to P^- and then guarding $L(P^-)$ by staying on $L(P^-)$ and moving toward the evader's projection.

Figure 5 illustrates the states in the overall strategy.

In the next section we show that:

1) The guarding strategy preserves the pursuer's progress. In other words, suppose the distance between Q and

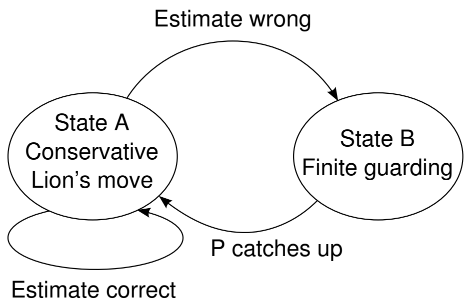

Fig. 5. Two-state pursuer strategy. The pursuer makes progress whenever he is in State A. When in State B, he guards his previous progress and goes back to State A in a finite number of steps.

P^- (the "correct" pursuer location) when the pursuer switches to the guarding mode is d. We will show that when the pursuer returns to the Lion's strategy with respect to Q, his distance to Q will be at least d.

2) In going from the lion's game to the guarding strategy and back to the lion's game, the guarding phase takes a finite number of steps.

In the next section, we will show that this pursuit strategy yields capture.

IV. ANALYSIS

The pursuer starts by computing the circle center Q and proceeds with the Lion's strategy using the conservative estimate. Suppose the pursuer is at P and the evader at E. The evader moves to a point on E^*E^- and the pursuer moves to the point P^* as described in our conservative Lion's move. Suppose that the evader is not at E^*. Then, the evader, P^* and Q are not collinear. This triggers the guarding phase.

Lemma 2: Suppose, after the pursuer's move, P, E and Q are not collinear. Let $d = |P^-Q|$ when this happens where P^- is the true lion's move corresponding to the evader move. There exists a pursuer strategy which guarantees that one of the following happens in a finite number of steps:

(i) P, E and Q are collinear and $d(P, Q) \geq d$.

(ii) $|PE| \leq 1$.

Proof: Let $L(P^-)$ be the tangent at P^- to the circle C centered at Q, passing through P^- (See Figure 4). Note that $L(P^-)$ touches both of the axes (since C also does).

Let the radius of C be $d = |QP^-|$. At the beginning of the guarding phase, the pursuer observes the evader's (conservative) move to, say, E'. Let x be the intersection of $L(P^-)$ with the line segment QE'. We refer to x as *the evader's projection onto* $L(P^-)$. If the pursuer can reach the projection in the first step, condition (i) holds and we are done.

Otherwise, the pursuer starts guarding $L(P^-)$ by moving onto the point on $L(P^-)$ that is closest to the evader's projection onto $L(P^-)$. Clearly, the distance between the pursuer's location and the evader's projection is bounded by $\delta = |P^*P^-|$. The guarding phase ends if the pursuer can move to the evader's projection.

During the guarding phase, the evader is inside the area bounded by one or both of the coordinate axes, the ray $QP(t)$ from the center Q passing through the pursuer's current location, and the line $L(P^-)$. As the pursuer guards the line $L(P^-)$, notice that the ray $QP(t)$ is rotating toward the ray $QE(t)$. Thus, before the pursuer hits the axis its moving toward, it is guaranteed that these two rays will cross (unless the evader crosses $L(P^-)$ first). When the rays cross, the pursuer can simply move on to the evader's projection. Since the pursuer has stayed on $L(P^-)$, he is outside circle C centered at Q passing through P^- i.e. at some finite time t', $|QP(t')| \geq d$ and the pursuer can resume with the Lion's strategy without loss of progress. Therefore, as long as the evader does not attempt to cross $L(P^-)$, the event described in part (i) of the lemma will happen.

In the remaining case, the evader crosses the line $L(P^-)$. Figure 6 illustrates such a case. Suppose the evader was at

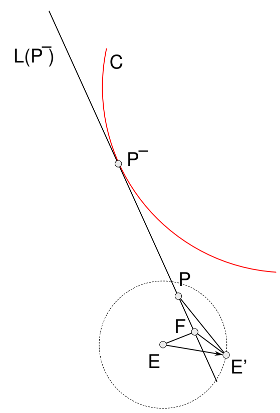

Fig. 6. If the evader E crosses the line $L(P^-)$, then the pursuer P moves toward E' and the final distance between the players is at most $\delta = |P^*P^-|$.

E. His projection F on the line $L(P^-)$ is a distance at most $\delta = |P^*P^-|$ away from P i.e. $|PF| \leq \delta$. The evader crosses $L(P^-)$ and lands at E' such that $|EE'| \leq 1$ (maximum step-size). Since the angle $\angle EFE'$ is greater than $\pi/2$ (the evader crossed), and $|EE'| = 1$, we have $|FE'| \leq 1$. Apply triangular inequality in $\Delta FE'P$ and we get

$$|PE'| \leq |PF| + |FE'| \leq \delta + 1$$

Now, the pursuer moves along PE' a unit distance to a point, call it P'. Then $|P'E'| = |PE'| - 1 \leq \delta$. Thus, soon after the evader crosses, the distance between the players is at most δ. ■

We now bound δ, the distance between points P^* and P^-. Let P be the pursuer location before it moved to P^*. By the definition of the lion's move, we have the angle $\angle P^-PP^* \leq \pi/2$ which means that $\delta \leq \sqrt{2}$. In fact, a tighter bound is possible, given by the next lemma.

Lemma 3: Let δ be the distance between the point P^* and P^- as explained in Lemma 2. $\delta \leq 1$.

Lemma 3 is proven in the appendix.

The following theorem gives us our main result.

Theorem 1: If the capture threshold is at least one, a pursuer with a bearing-only sensor can capture the evader in a finite number of steps by following the two-state pursuer strategy.

Proof: Whenever the pursuer makes the Lion's move, he makes a definite progress as explained by Sgall in [3]. In our strategy, the pursuer plays a single Lion's game with finite capture time. However, after a pursuer move, the Lion's move may not exist. We showed that whenever this happens, either (i) the pursuer returns to the original Lion's game in a finite number of steps and claims the progress possible with the correct lion's move, or (ii) the distance between the players drops below one and hence, the evader is captured if the capture threshold is at least one.

The whole game can now be viewed as a finite sequence of games $G_1, G_2, G_3, \ldots, G_n$ where games with odd indices are parts of a single Lion's game (with increasing progress) and the even indices are guarding games that last a finite number of steps and preserve the pursuer's progress from one Lion's game to the next.

We now provide an upper-bound on the capture time. Suppose the game starts with the pursuer at (x_0, y_0) and the evader at (x'_0, y'_0) and let $\alpha_0 = (y_0 - y'_0)/(x_0 - x'_0)$ be the initial slope of the line joining them. Then, the total capture time for the Lion's game (sum of the times for $G_1, G_3, G_5, \ldots$) as derived in [3] is:

$$\begin{aligned} T_L &= max\{(x_0 + y_0(\alpha_0 + \sqrt{1+\alpha_0{}^2}))^2, \\ &\quad (y_0 + x_0(\alpha_0{}^{-1} + \sqrt{1+\alpha_0{}^{-2}}))^2 \end{aligned}$$

A single guarding game lasts for a time of at most the maximum of the X and Y coordinates of the pursuer at P^*, which, by construction of the initial circle centered Q and the imposition of Invariant 1, is bounded from above by the maximum of the X and Y coordinates of the center Q. Thus the capture time for a single guarding game is given by

$$\begin{aligned} T_G &= max\{x_0 + y_0(\alpha_0 + \sqrt{1+\alpha_0{}^2}), \\ &\quad x_0 + x_0(\alpha_0{}^{-2}(1 + \sqrt{1+\alpha_0{}^2}))\} \end{aligned}$$

Since, in the worst case, the switch from the Lion's game to a guarding game happens at the end of each time step, the total capture time is bounded by $T_L T_G$. ■

A. Knowledge about the evader's initial location

We started the paper with the assumption that the pursuer knows the exact initial location of the evader. In this section, we remove this assumption. The main idea is to have the pursuer perform a "safe" initial move and obtain the evader's position by triangulation. This is formalized in the following lemma.

Lemma 4: Call the two-state pursuer strategy described earlier as S_p. As explained in Section III-B, S_p requires the initial location of the evader for our analysis to hold. If the pursuer does not know the initial location of the evader, then there exists an initial pursuer move that allows him to obtain the exact evader location, after which he can continue with S_p.

Proof: Suppose the pursuer starts at P and the evader at E such that the coordinates of E lie in between the coordinates of P and the origin of the first quadrant. Further, suppose the evader moves to E' and the pursuer has the bearing ray through E'. Call this ray $r(P)$. Our idea is for the pursuer to move to a point P' and then obtain his sensing ray $r(P')$. The intersection of $r(P)$ and $r(P')$ gives the pursuer the exact location of E' and now he continues with S_p.

Note that P' has to be chosen in such a way that Invariant 1 is not broken and the rays $r(P)$ and $r(P')$ do not coincide. There are two cases that arise.

Suppose that E' lies between the pursuer and the origin, then the pursuer can simply move parallel to one of the axes, toward the evader and the new sensing ray will give him the required information. Invariant 1 is clearly preserved.

In the event that E' crosses the pursuer along one of his coordinates, say Y (the other coordinate follows a symmetric argument), the pursuer moves one unit away from the origin parallel to the Y-axis. This guarantees that Invariant 1 is preserved because they initially started with a positive Y-separation. Further, the intersection of the new sensing ray with the previous one gives the pursuer the exact location of the evader and he plays the rest of the game following our strategy S_p. ■

V. Conclusion

In this paper, we studied the effect of a common sensing limitation on a well-known pursuit-evasion game. We showed that, a pursuer with a bearing-only sensor can decrease the distance between the players to the step-size. A remaining open question is whether the pursuer can decrease the distance to zero.

Throughout the paper, we assumed that the pursuer can localize itself precisely. Another direction of future research is to incorporate uncertainties regarding the pursuer's location into the pursuit strategy.

VI. ACKNOWLEDGMENTS

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APPENDIX

PROOF OF LEMMA 3.

We derive an upper bound on $\delta = |P^*P^-|$ with geometric arguments.

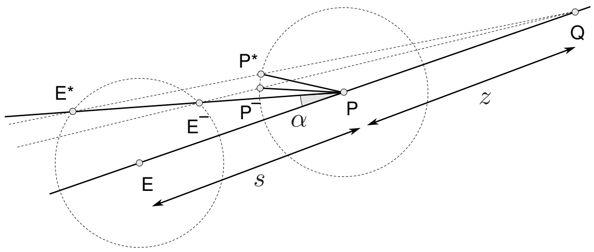

Fig. 7. Upper bound on $|P^*P^-|$: We express $|P^*P^-|$ as a function of three parameters s, z and α.

We express the angle $\angle P^-PP^*$ as a function of three parameters, as shown in Figure 7: (i) s, the distance between the evader E and the pursuer P before the move, (ii) z, the distance between the center Q of the Lion's circle and the pursuer P and (iii) α, the angle of the sensing ray of the pursuer with the line PE. By studying how δ varies with each parameter, we compute an upper bound on the distance between P^* and P^-. Observe that since both of these points lie on a unit circle centered at P, $|P^*P^-|$ and $\angle P^-PP^*$ are directly related. For the rest of our analysis, we refer to this parametrized angle as $\beta(s, \alpha, z)$.

Effect of z. As the point Q moves away from P (i.e. as z goes to infinity), observe that P^* rotates about the unit circle centered at P at a rate quicker than P^- does. Thus $\beta(s, \alpha, z)$ increases as $z \longrightarrow \infty$. This means that the highest value of β is achieved when Q is at infinity. In this case, the lines QP^-E^- and QP^*E^* become parallel to QPE. To minimize β, we can focus on the remaining two parameters and set z to ∞.

Effect of s.

Lemma 5: Let $2 \leq s_2 < s_1$ be two evader locations and α_1 be a viewing angle. There exists a viewing angle α_2 such that $\beta(s_2, \alpha_2) > \beta(s_1, \alpha_1)$.

Proof:

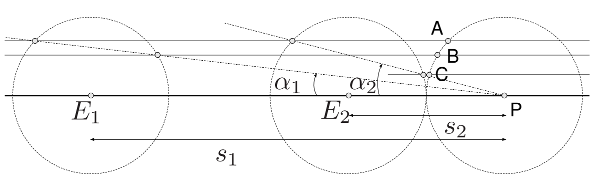

Fig. 8. Dominance of the s parameter: a lower value of s gives a higher value of β, although for a different value of α.

Consider the situation shown in Figure 8. Compare the two configurations P, E_1 and P, E_2 such that $s_1 > s_2$. For the evader at E_1, the lines from Q intersect the pursuer's unit circle at A and B. For any configuration with the evader at E_2 such that $s_2 < s_1$, we can adjust α_2 to have the same top point A on the pursuer's unit circle. Observe that this ray intersects the unit circle centered at E_2 at a point lower than B by construction, because the unit circle centered at E_2 is closer to P. Thus the rays from Q through these points intersect the pursuer's unit circle at A and C, where C is farther from A than B. Since A, B and C are on the same circle centered at P, we get $\angle BPA < \angle CPA$ i.e. $\beta(s_1, \alpha_1) < \beta(s_2, \alpha_2)$. Thus parameter s is such that smaller values dominate larger values. ∎

Since $s > 1$ (otherwise the evader is already captured), Lemma 5 allows us to restrict our search for maximum β to the interval $s \in [1, 2]$.

When s goes below 2, the unit circles centered at the evader and pursuer intersect, causing the point on the evader estimate closest to the pursuer to move from being on the evader's circle to being on the pursuer's circle. This made it difficult to obtain an analytical solution. We derived the symbolic expression for β and solved this problem numerically by plotting s versus α versus β in MATLAB. Figure 9 shows the 3D surface plot.

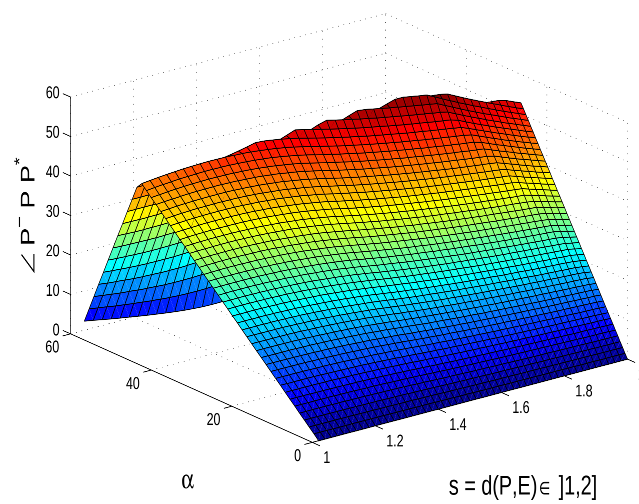

Fig. 9. Plot of $\beta = \angle P^-PP^*$ versus parameters s and α.

As it can be seen from the figure, the maximum value attained is less than $\pi/3$. Since $|P^-P|$ and $|P^*P|$ are both of length 1, $|P^*P^-|$ is bounded from above by 1.