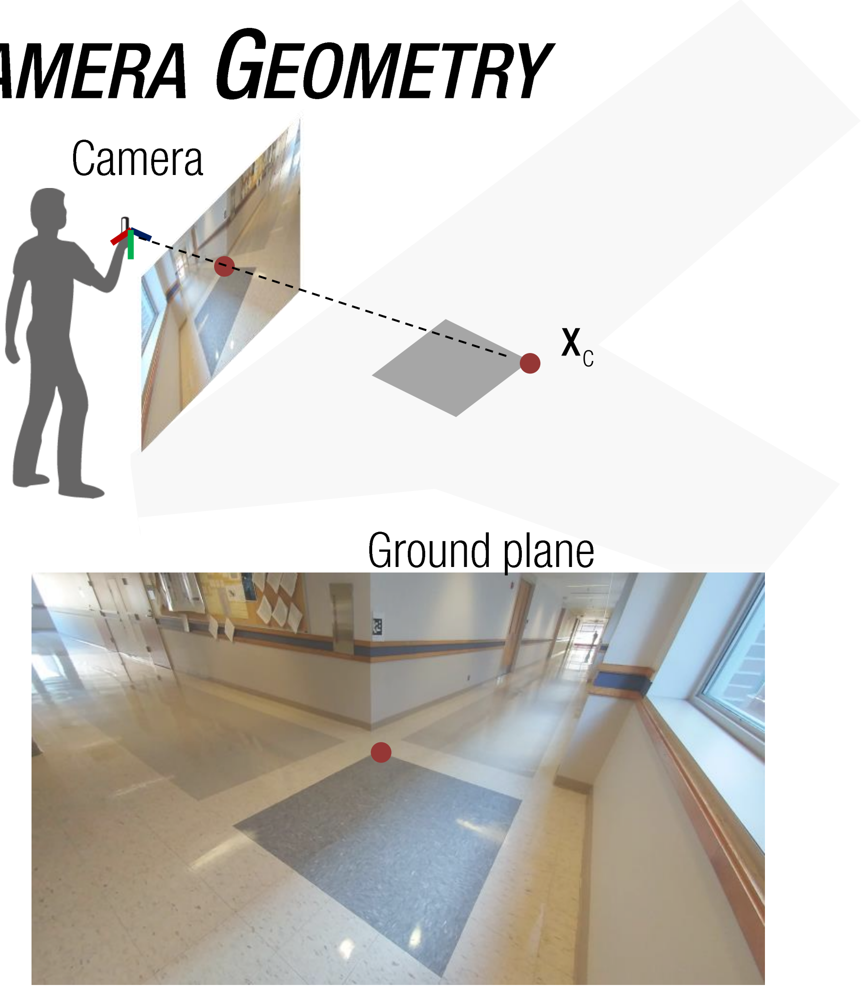
Projection Matrix

Hyun Soo Park

Camera Geometry

CAMERA GEOMETRY

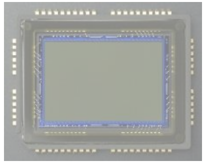
CAMERA GEOMETRY



Recall camera projection matrix:

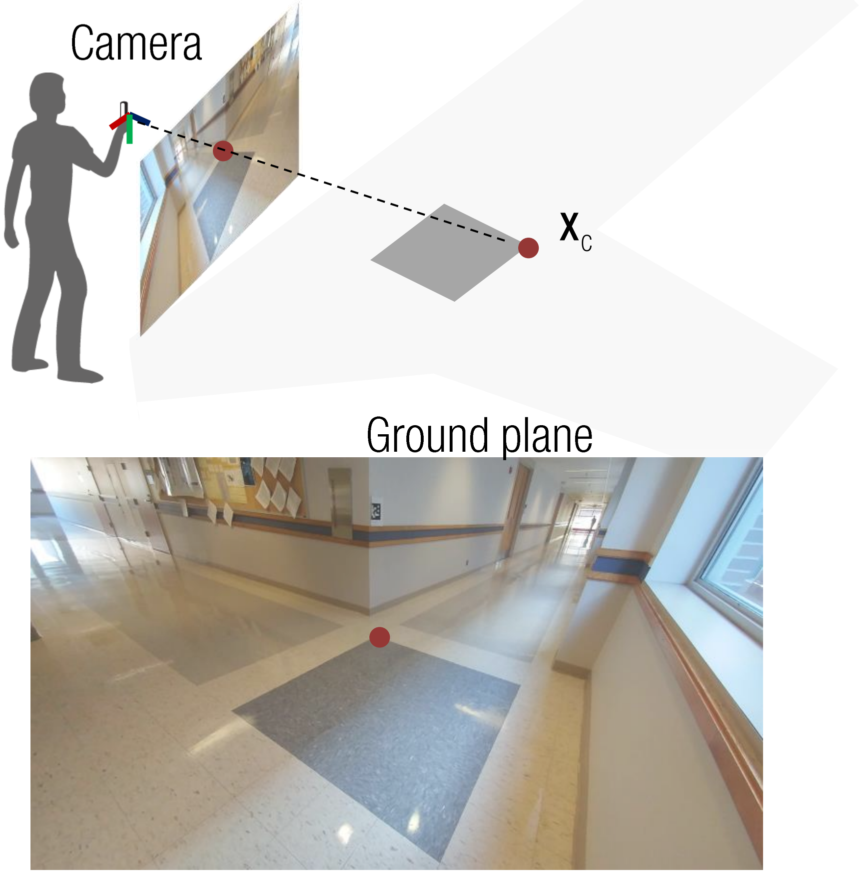
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

+



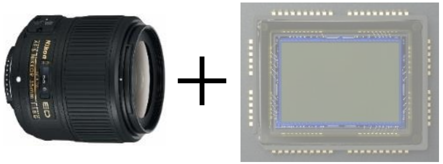
Camera intrinsic parameter
: metric space to pixel space

CAMERA GEOMETRY



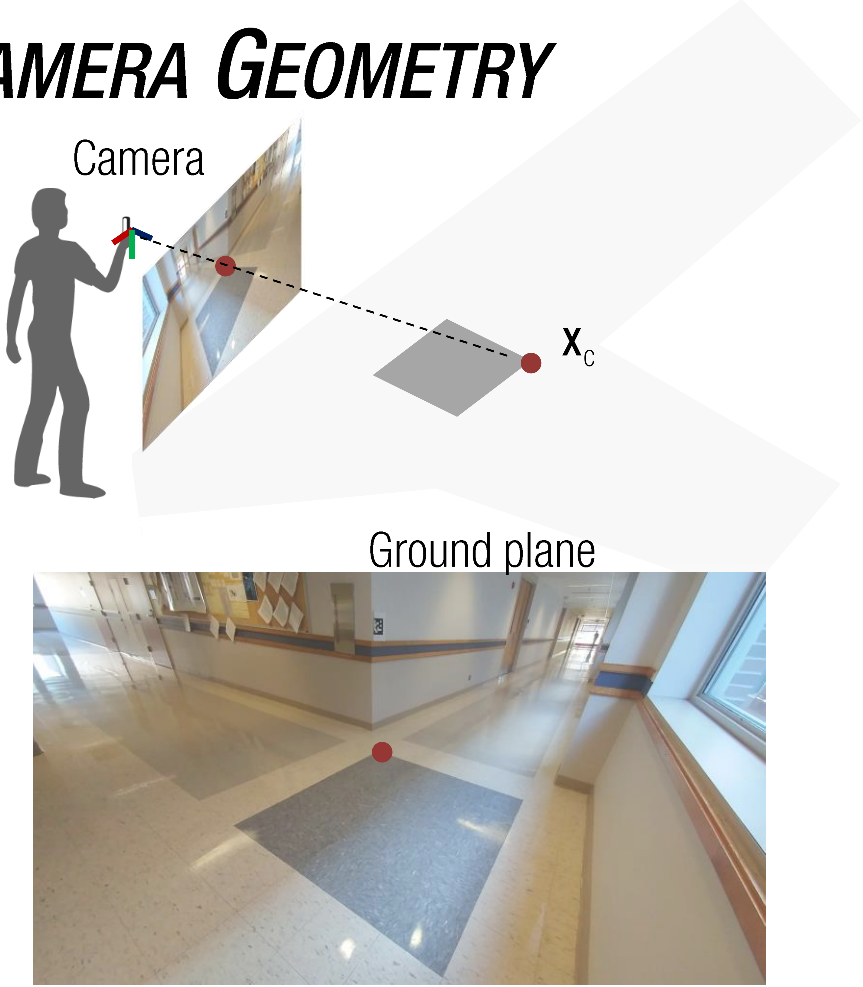
Recall camera projection matrix:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



Camera intrinsic parameter
: metric space to pixel space

Camera Geometry

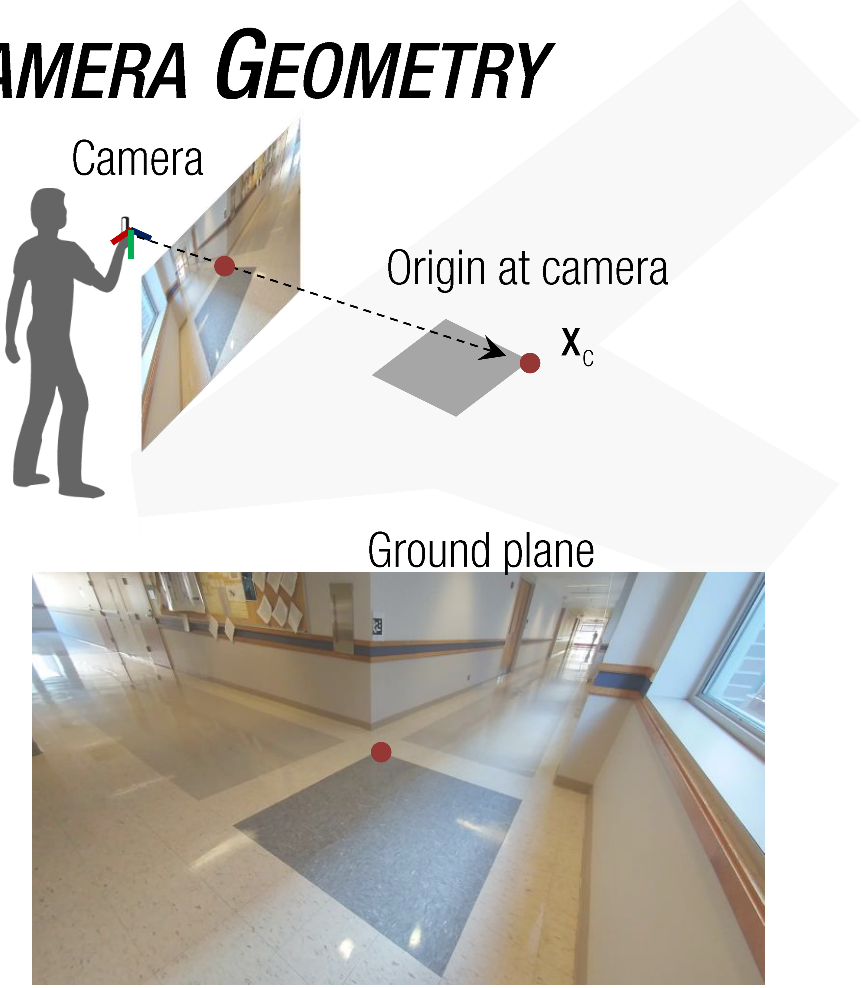


Recall camera projection matrix:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

2D image (pix) 3D world (metric)

Camera Geometry



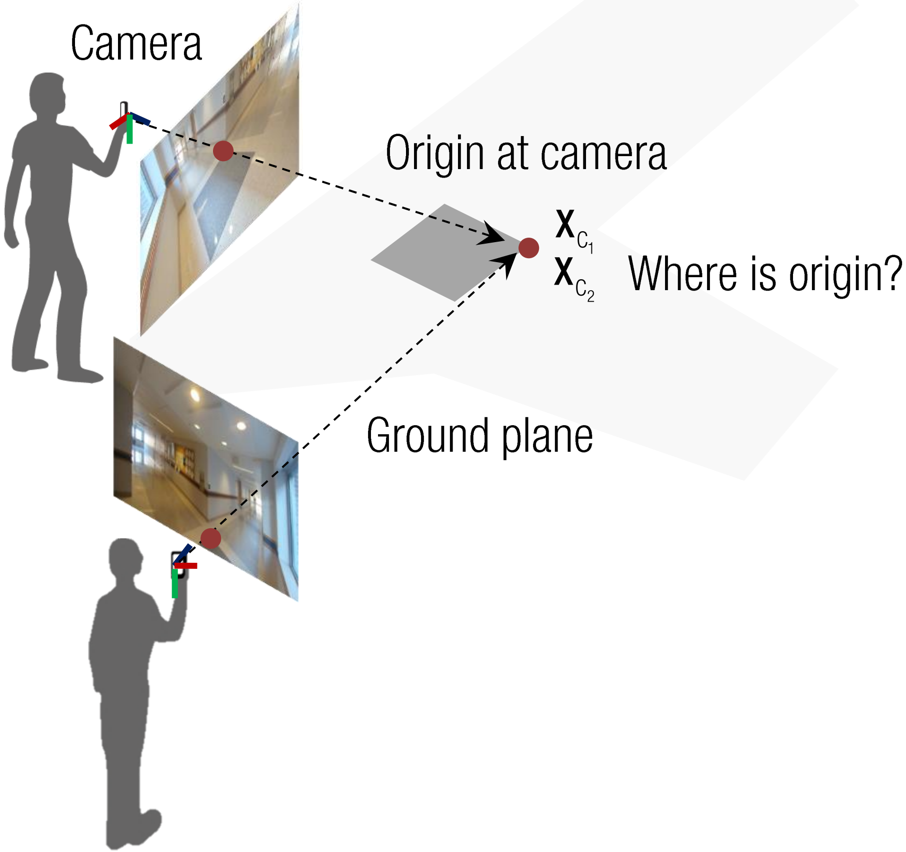
Recall camera projection matrix:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

2D image (pix) 3D world (metric)

$$\rightarrow \lambda \mathbf{K}^{-1} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{X}_C$$

CAMERA GEOMETRY



Recall camera projection matrix:

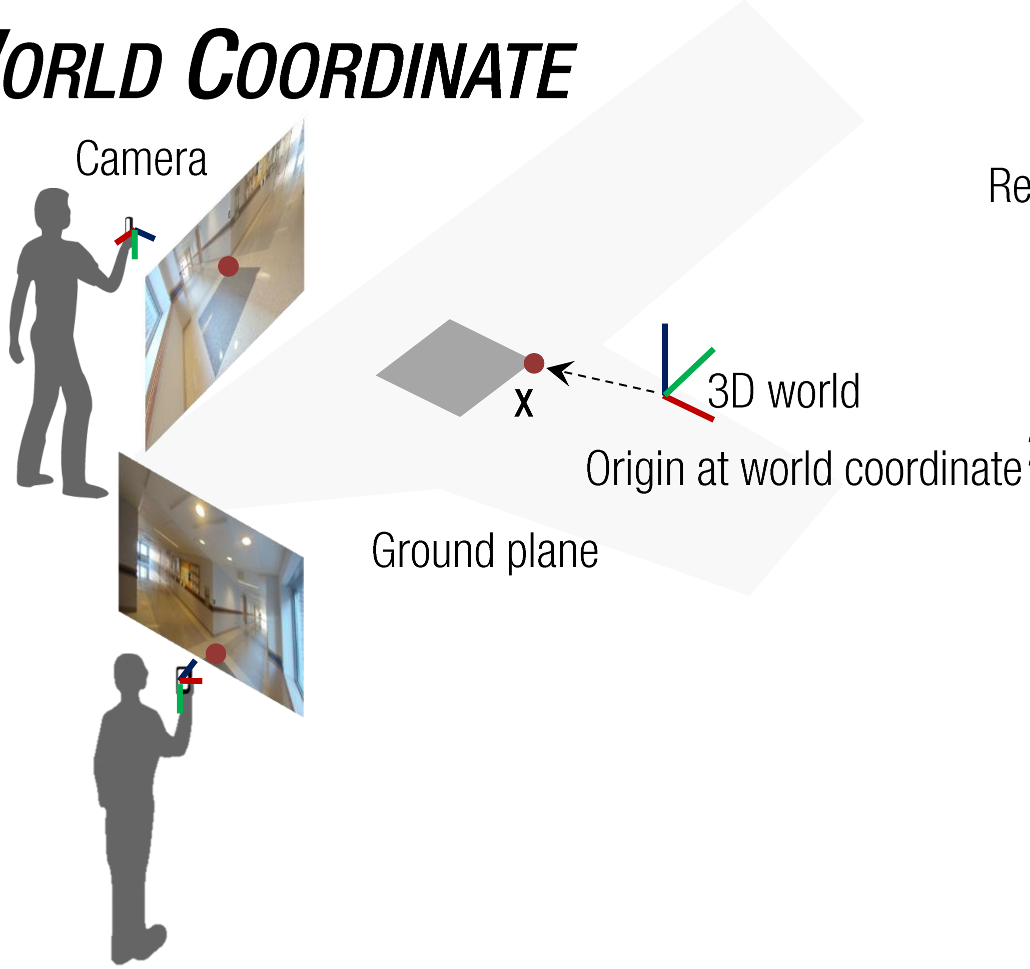
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

2D image (pix) 3D world (metric)

$$\longrightarrow \lambda \mathbf{K}^{-1} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \mathbf{X}_{C_1}$$

$$\lambda \mathbf{K}^{-1} \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix} = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \mathbf{X}_{C_2}$$

WORLD COORDINATE



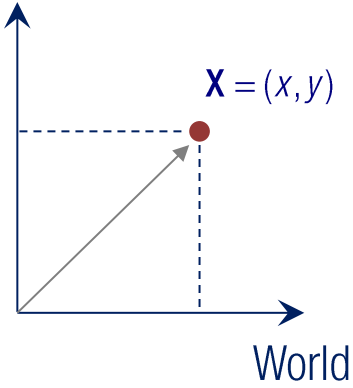
Recall camera projection matrix:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \mathbf{X}$$

2D image (pix) 3D world (metric)

Point Rotation

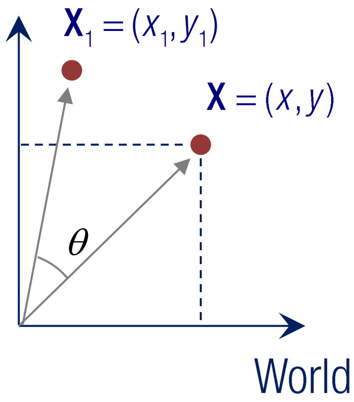
2D rotation



$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Point Rotation

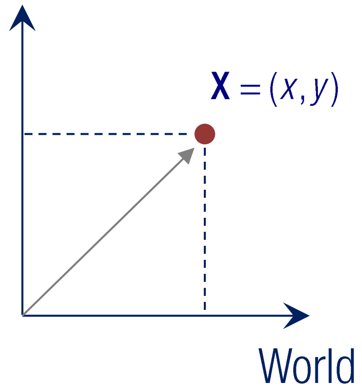
2D rotation



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate Transform (Rotation)

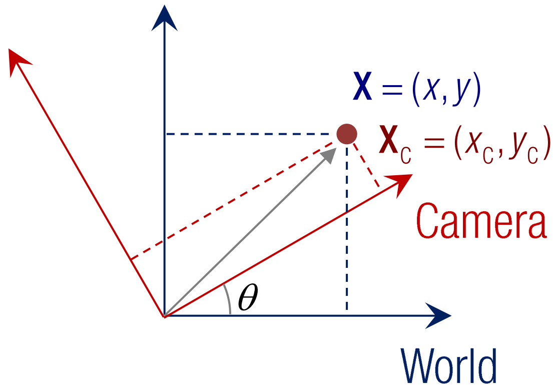
2D coordinate transform:



$$\begin{bmatrix} x \\ y \end{bmatrix}$$

COORDINATE TRANSFORM (ROTATION)

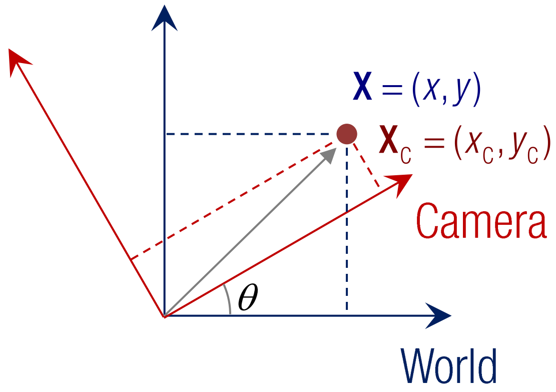
2D coordinate transform:



$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \quad ? \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate Transform (Rotation)

2D coordinate transform:

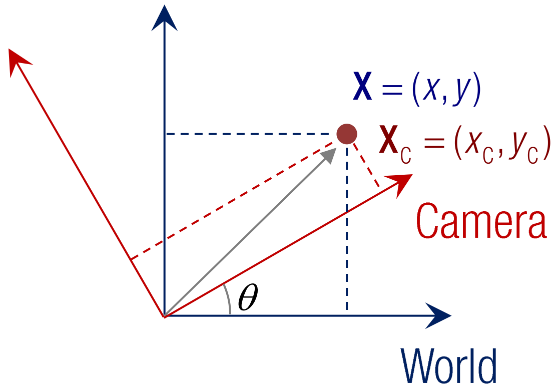


$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate transformation:
Inverse of point rotation

Coordinate Transform (Rotation)

2D coordinate transform:

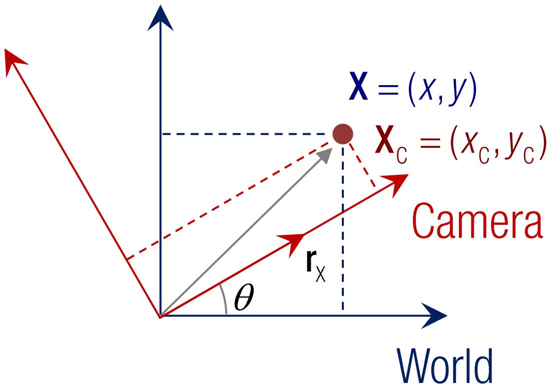


$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right) = \cos^2\theta + \sin^2\theta = 1$$

Coordinate Transform (Rotation)

2D coordinate transform:

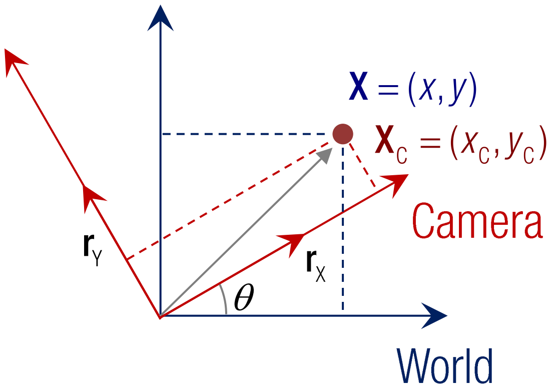


$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} \cos\theta \; \mathbf{r}_x & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\mathbf{r}_x$: x axis of camera seen from the world

Coordinate Transform (Rotation)

2D coordinate transform:



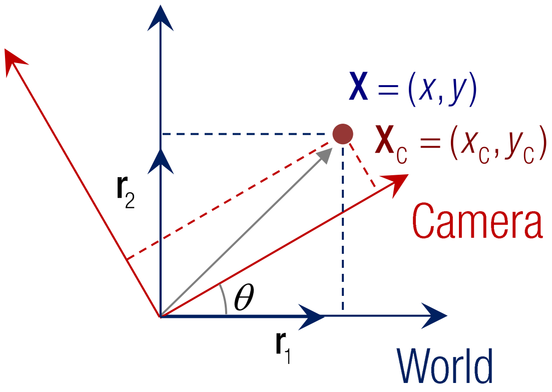
$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{matrix} \mathbf{r}_X \\ \mathbf{r}_Y \end{matrix}$$

$\mathbf{r}_X$: x axis of camera seen from the world

$\mathbf{r}_Y$: y axis of camera seen from the world

Coordinate Transform (Rotation)

2D coordinate transform:

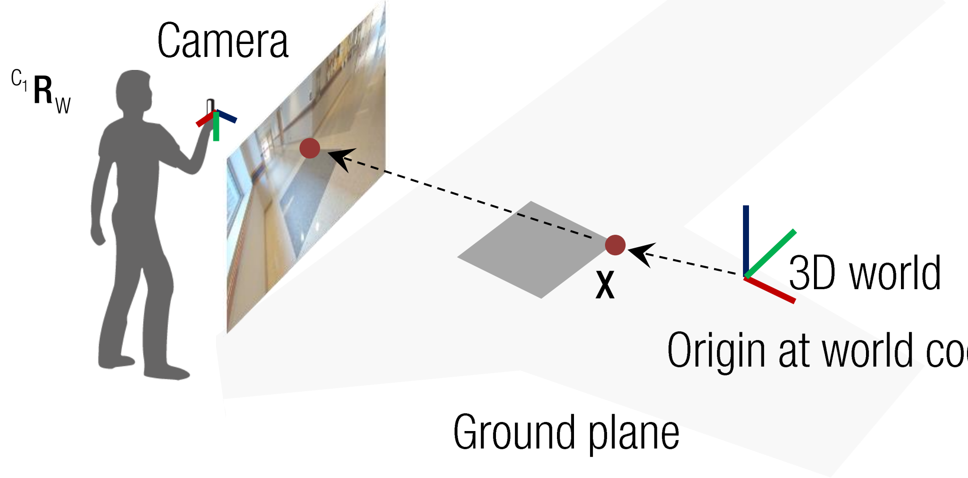


$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ \begin{array}{cc} \mathbf{r}_1 & \mathbf{r}_2 \end{array} \end{array} \begin{bmatrix} x \\ y \end{bmatrix}$$

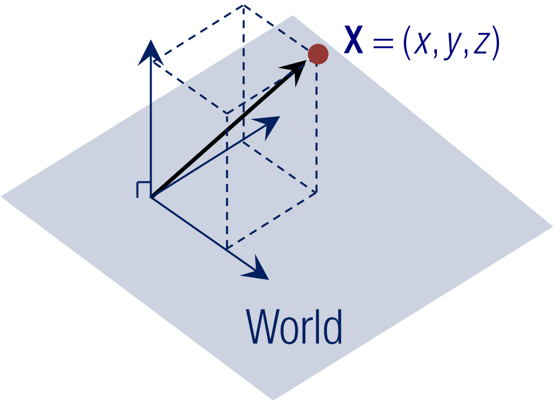
$\mathbf{r}_1$: x axis of world seen from the camera

$\mathbf{r}_2$: y axis of world seen from the camera

COORDINATE TRANSFORM (ROTATION)



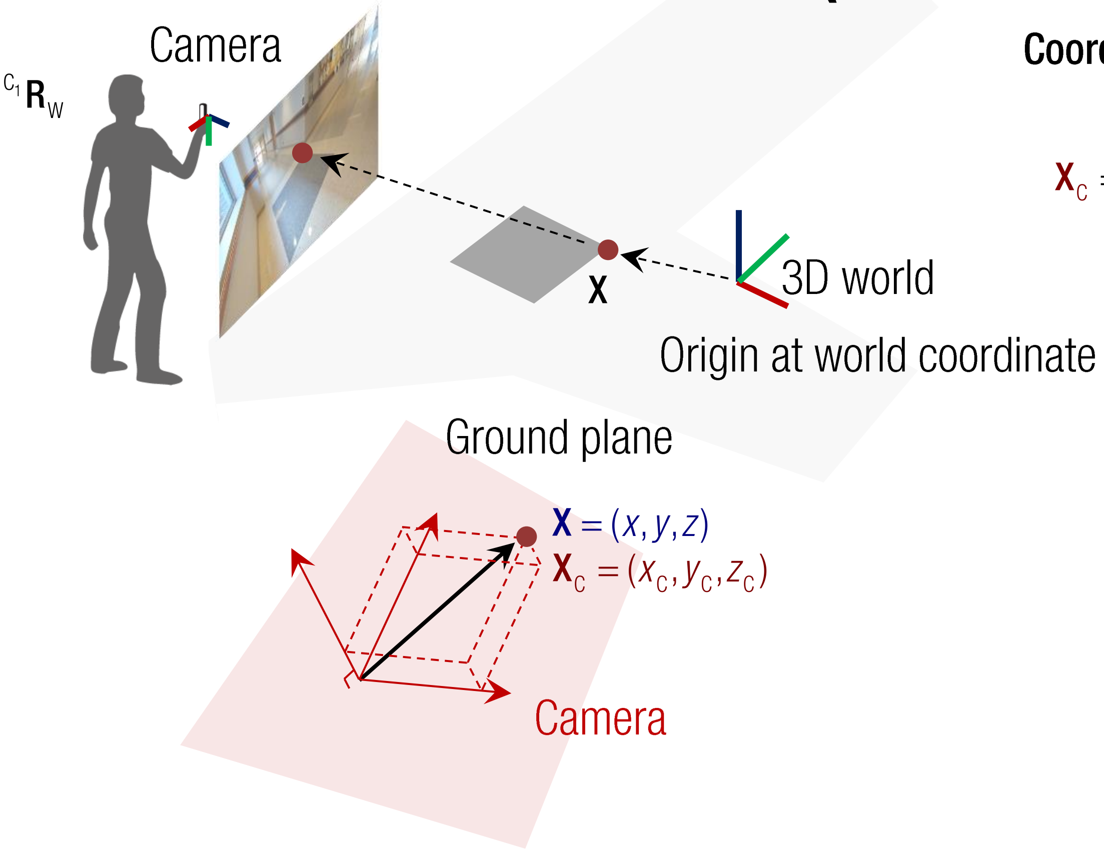
Ground plane



Coordinate transformation from world to camera:

$\mathbf{X}_C = \quad ? \quad \mathbf{X}$

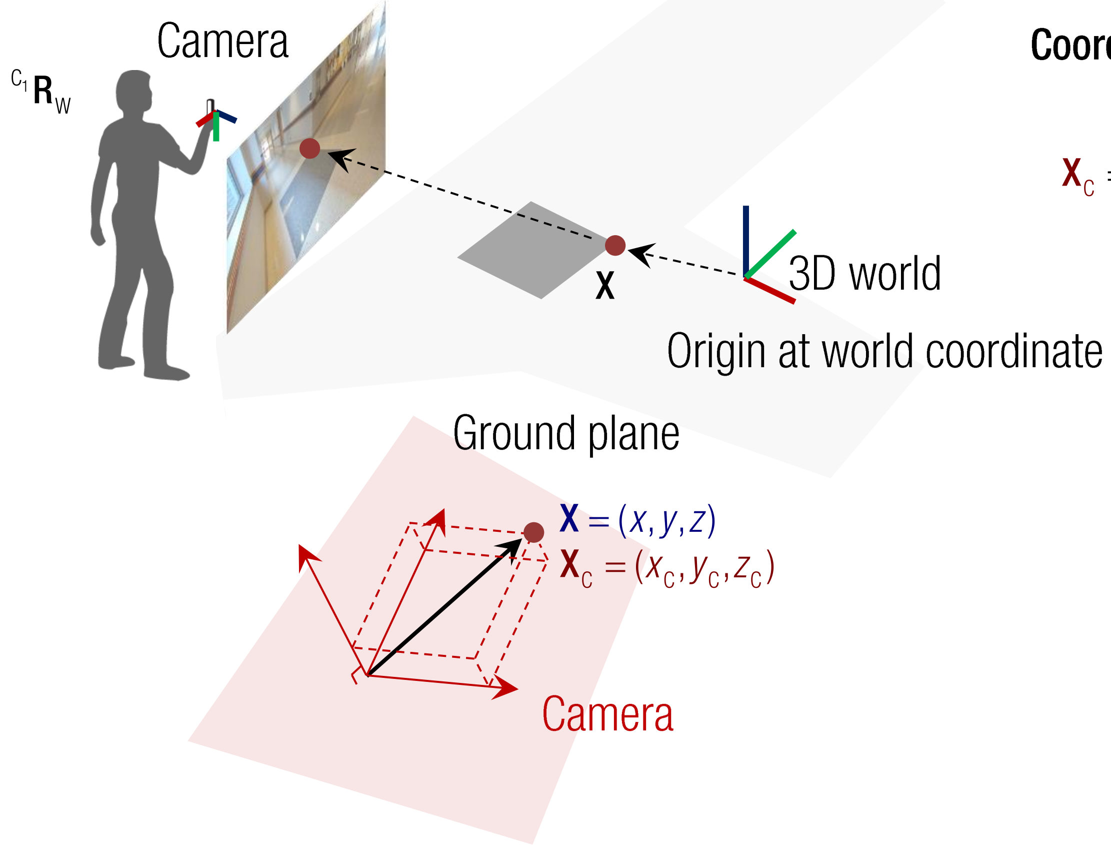
Coordinate Transform (Rotation)



Coordinate transformation from world to camera:

$$\mathbf{X}_C = \quad ? \quad \mathbf{X}$$

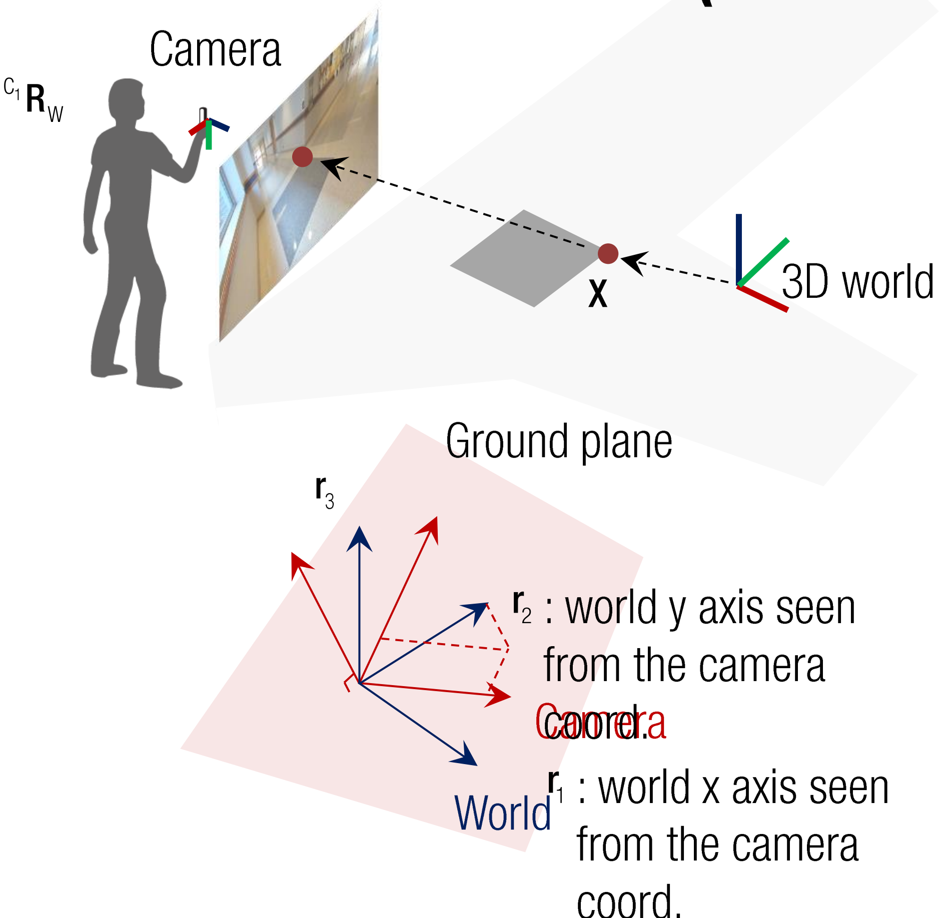
Coordinate Transform (Rotation)



Coordinate transformation from world to camera:

$$\mathbf{X}_C = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \mathbf{X} = {}^{C}\mathbf{R}_W \mathbf{X}$$

Camera Projection (Pure Rotation)



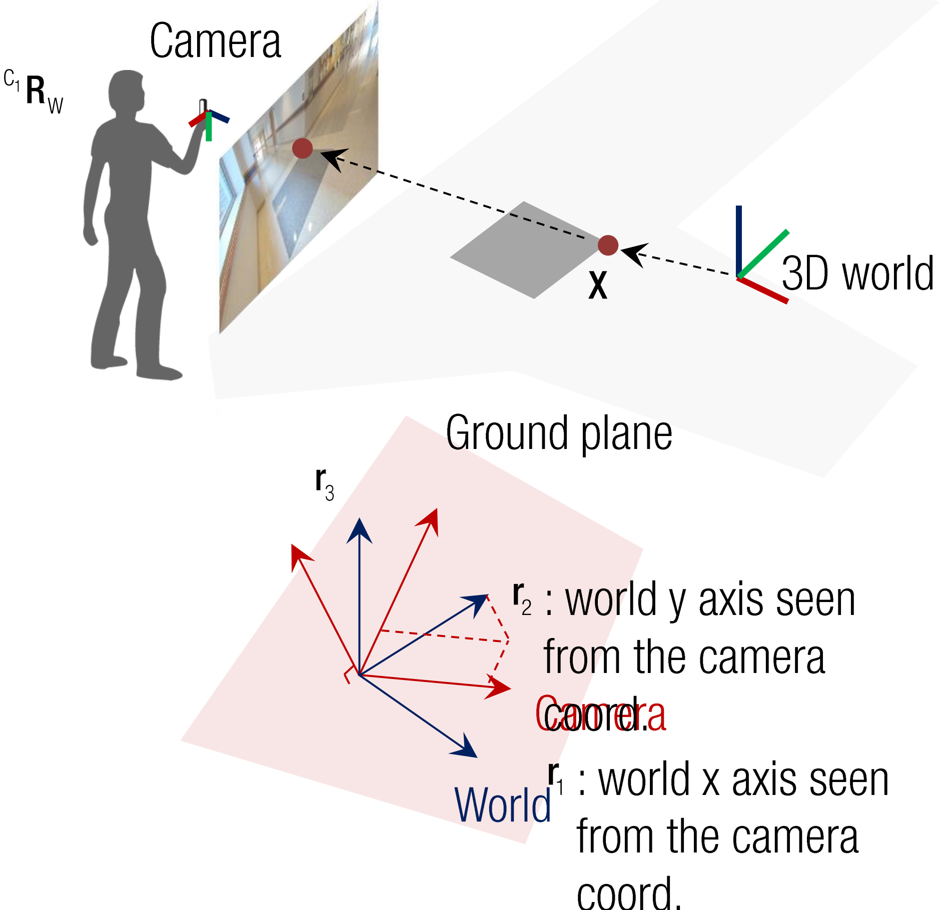
Coordinate transformation from world to camera:

$$\mathbf{X}_C = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \mathbf{X} = {}^{C}\mathbf{R}_W \mathbf{X}$$

Camera projection of world point:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

Camera Projection (Pure Rotation)



Coordinate transformation from world to camera:

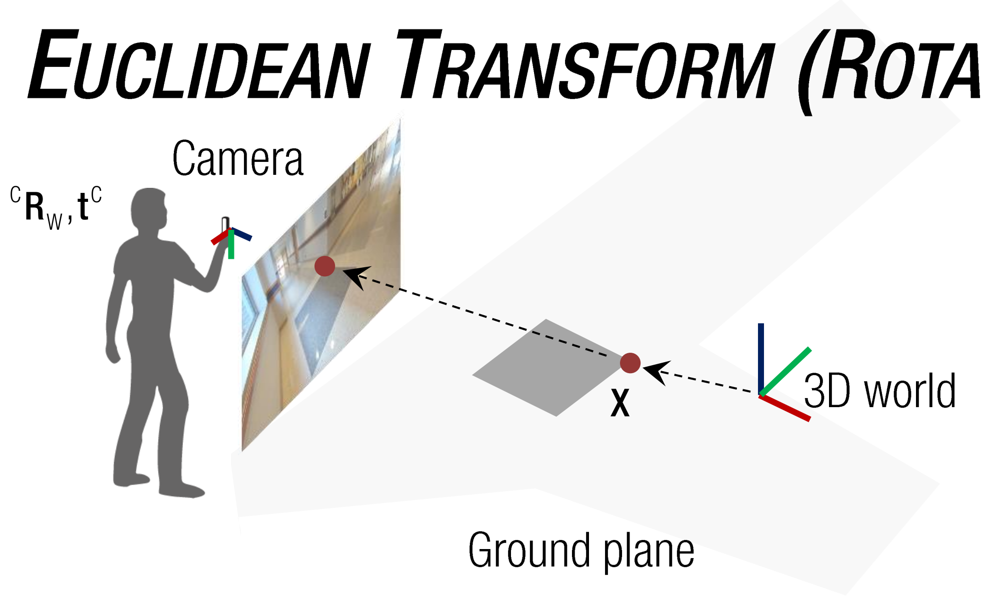
$$\mathbf{X}_C = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \mathbf{X} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \mathbf{X} = {}^{C}\mathbf{R}_W \mathbf{X}$$

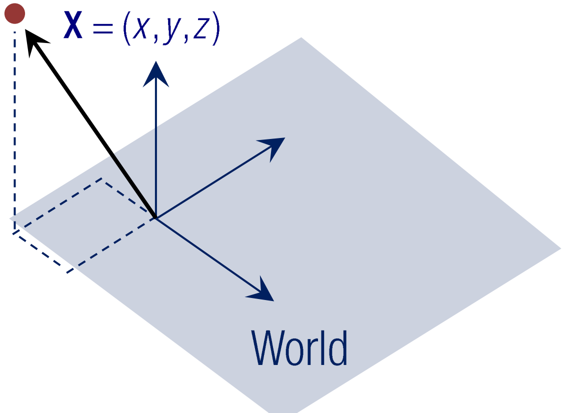
Camera projection of world point:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

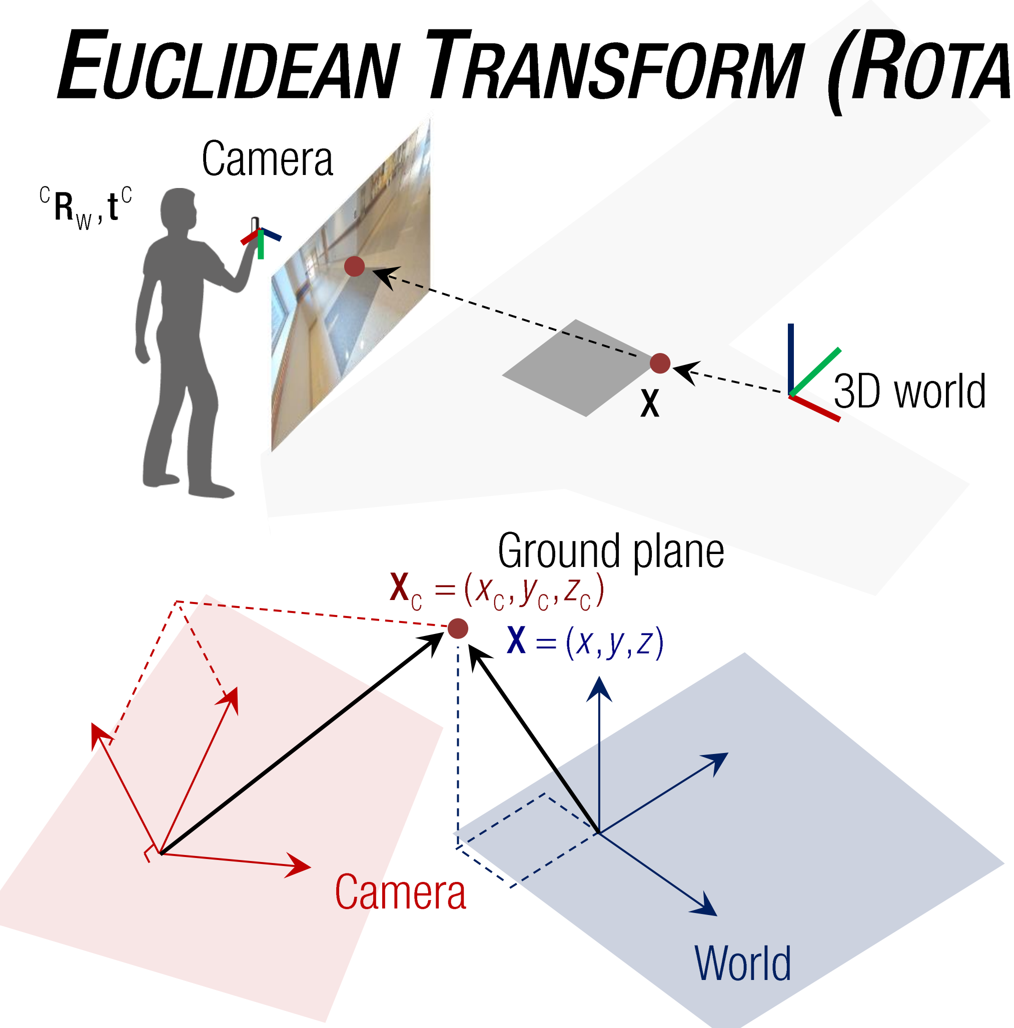
$$= \mathbf{K} \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Euclidean Transform (Rotation+Translation)



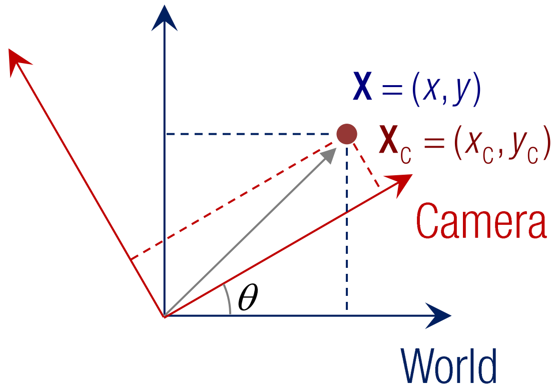


Euclidean Transform (Rotation+Translation)



EUCLIDEAN TRANSFORM (ROTATION+TRANSLATION)

2D coordinate transform:

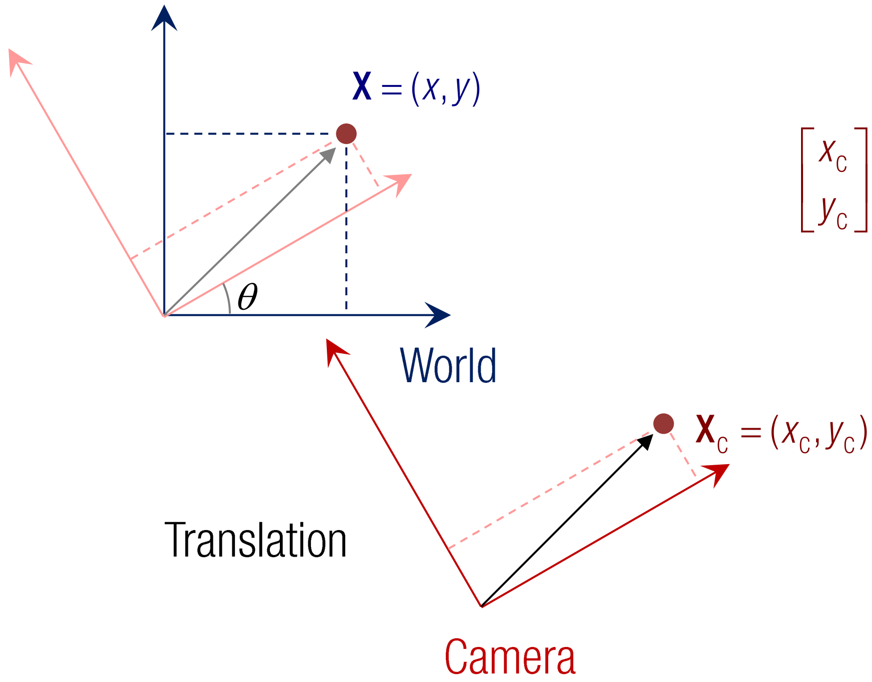


$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate transformation:
Inverse of point rotation

Euclidean Transform (Rotation+Translation)

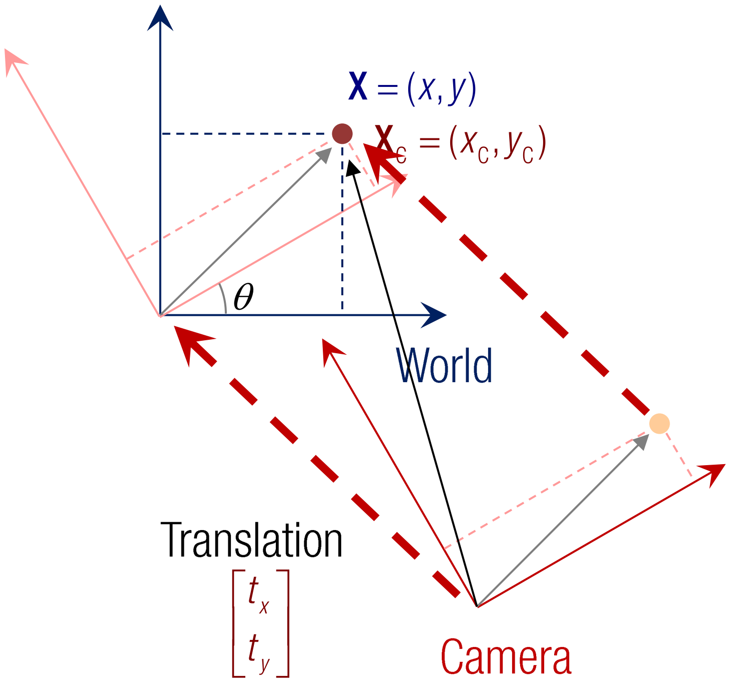
2D coordinate transform:



$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

EUCLIDEAN TRANSFORM (ROTATION+TRANSLATION)

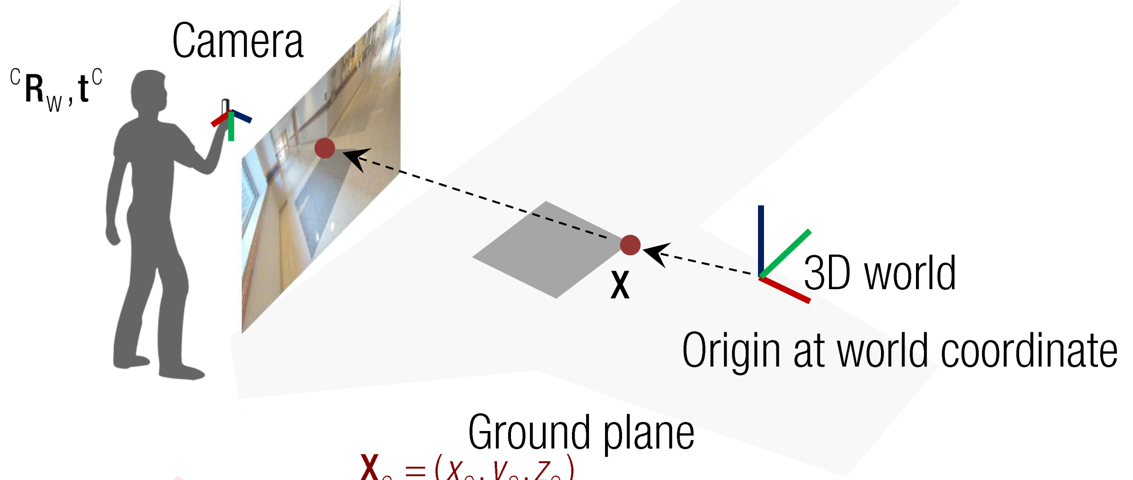
2D coordinate transform:

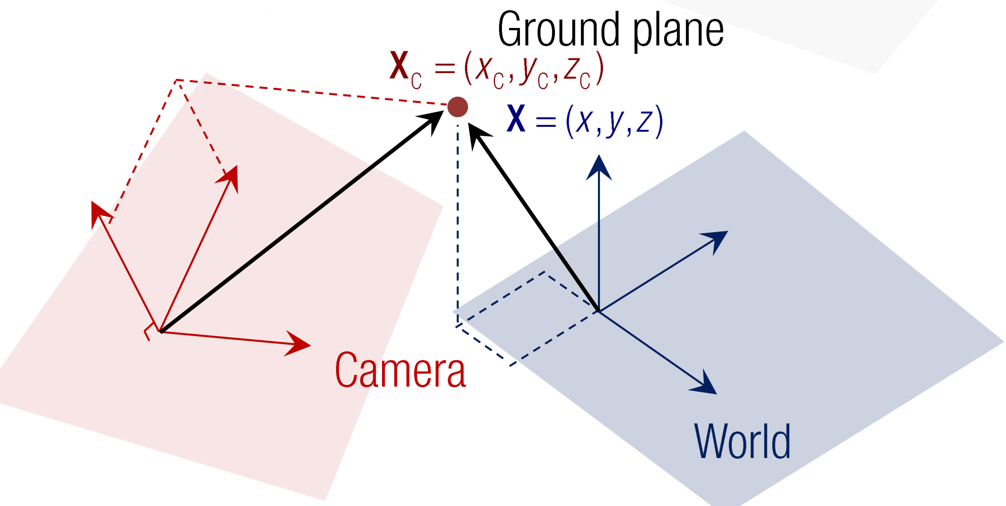


$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

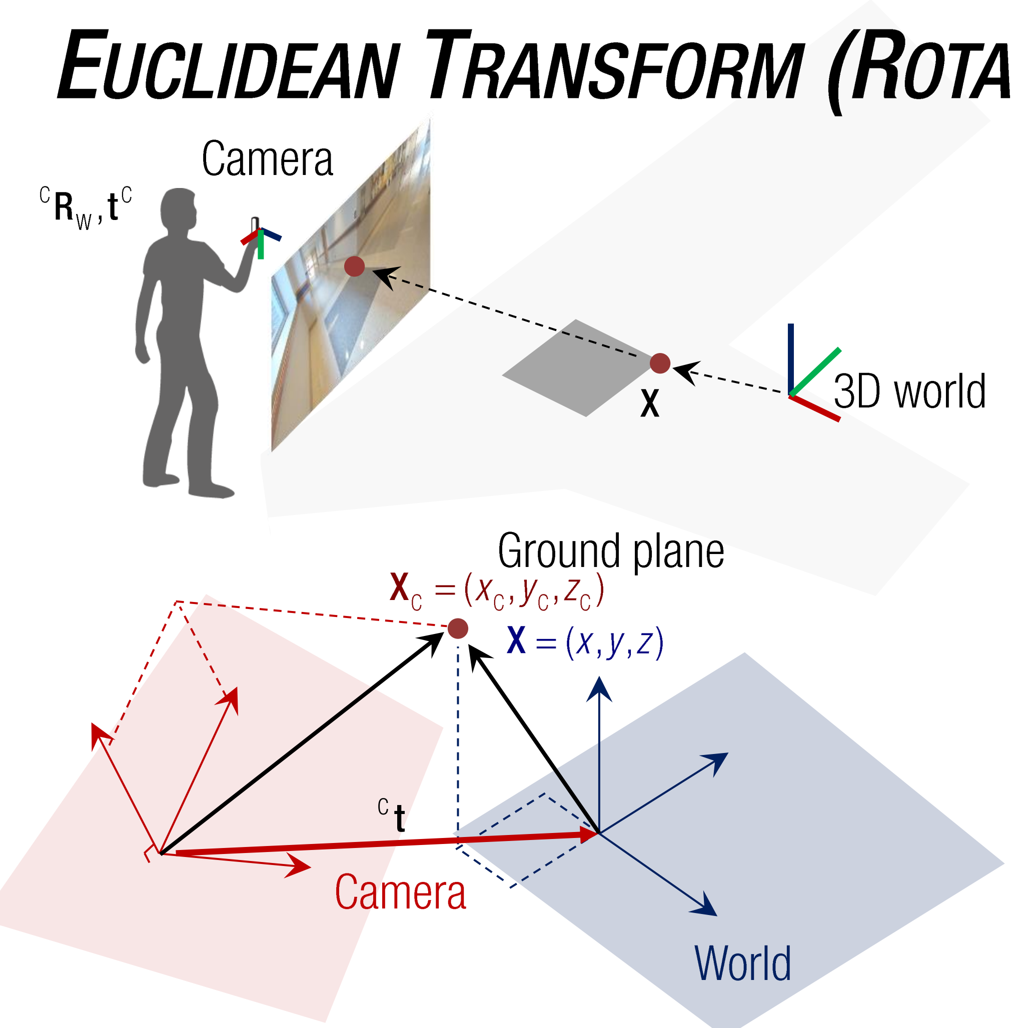
$\begin{bmatrix} t_x \\ t_y \end{bmatrix}$: the location of world coordinate seen from camera coord.

Euclidean Transform (Rotation+Translation)





EUCLIDEAN TRANSFORM (ROTATION+TRANSLATION)

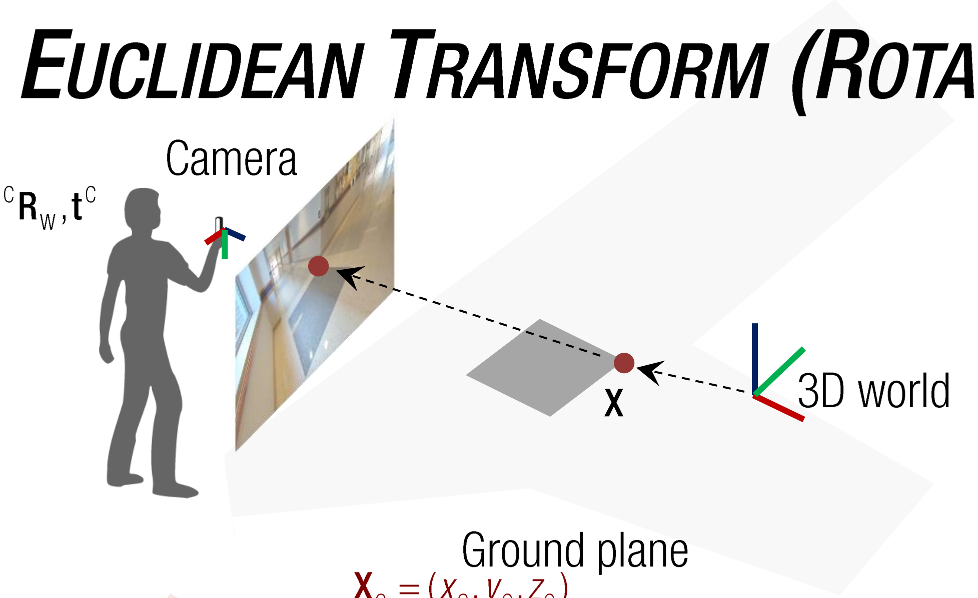


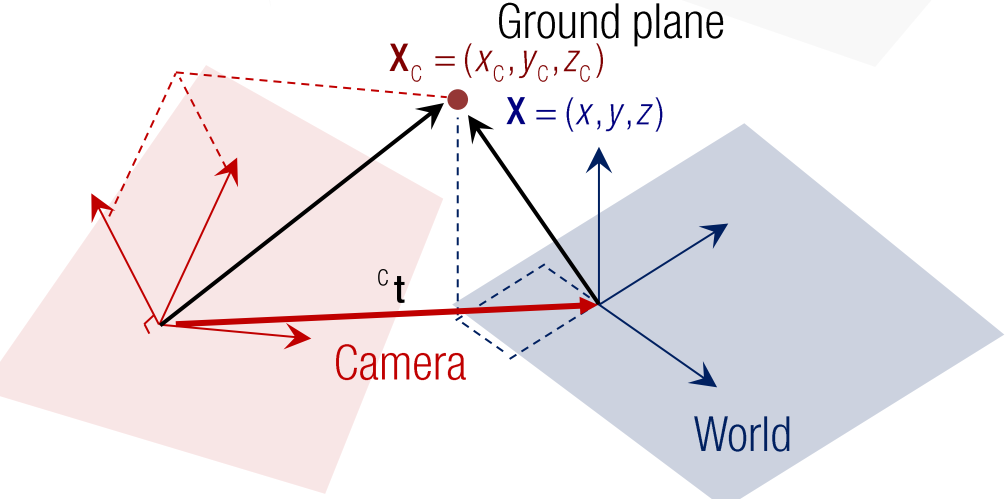
Coordinate transformation from world to camera:

$$\mathbf{X}_C = {}^{C}\mathbf{R}_W \mathbf{X} + {}^{C}\mathbf{t}$$

where ${}^{C}\mathbf{t}$ is the world orgin seen from camera.

Euclidean Transform (Rotation+Translation)



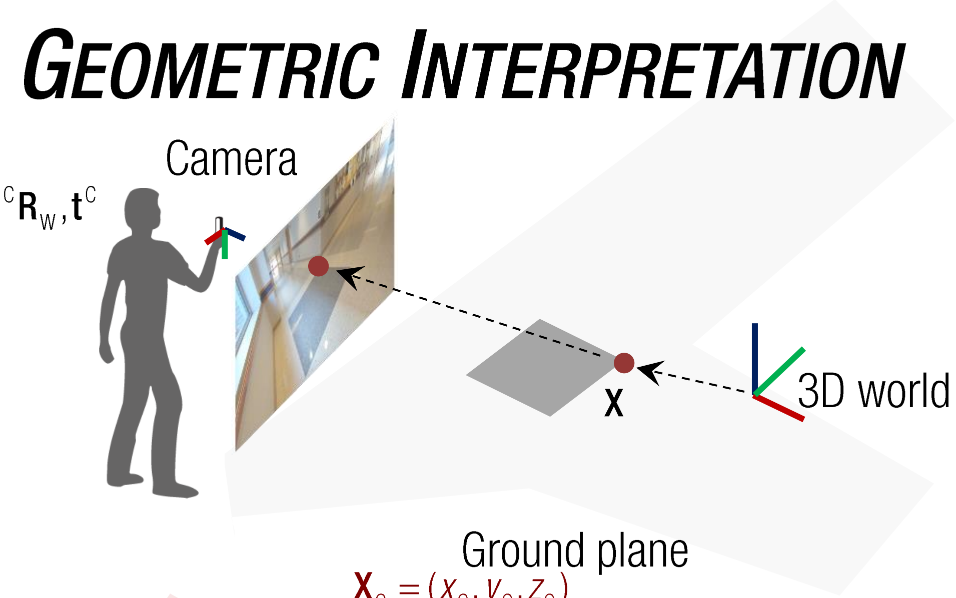


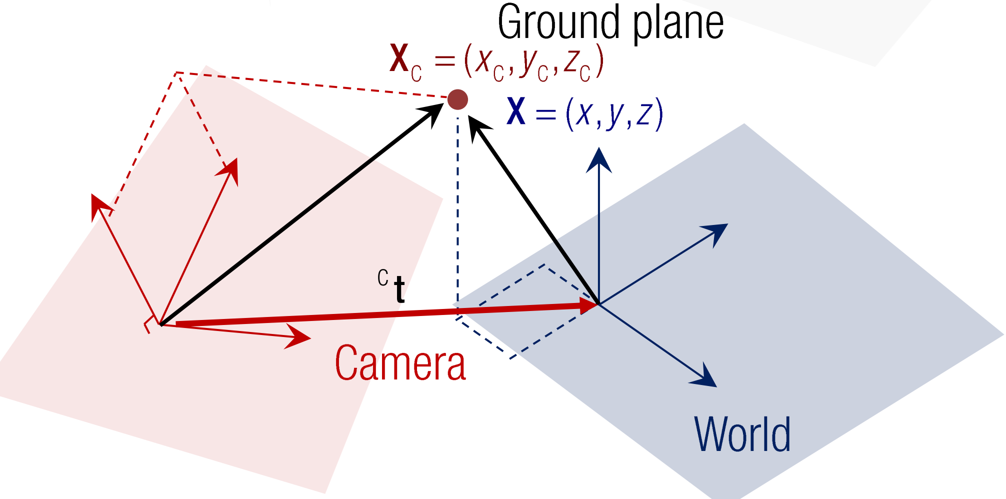
Coordinate transformation from world to camera:

$$\mathbf{X}_C = {}^{C}\mathbf{R}_W \mathbf{X} + {}^{C}\mathbf{t} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

where ${}^{C}\mathbf{t}$ is the world orgin seen from camera.

Geometric Interpretation





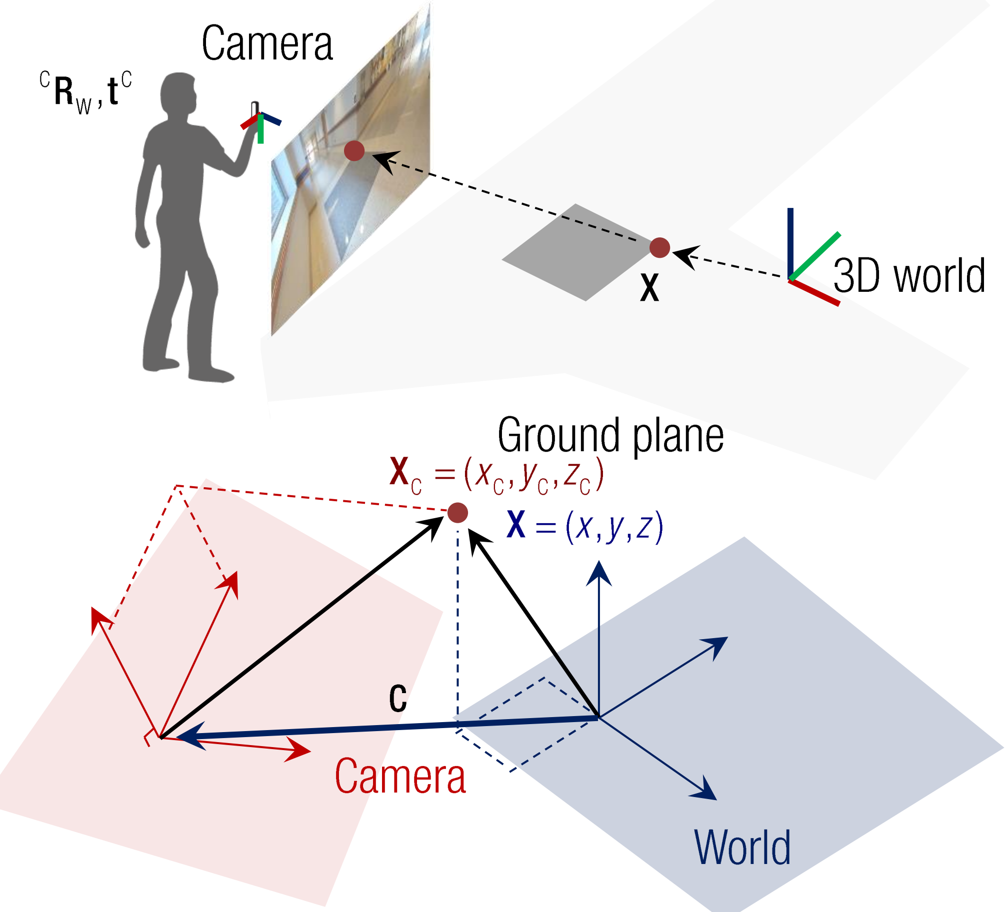
Coordinate transformation from world to camera:

$$\mathbf{X}_C = {}^{C}\mathbf{R}_W \mathbf{X} + {}^{C}\mathbf{t} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

where ${}^{C}\mathbf{t}$ is the world orgin seen from camera.

Rotate and then, translate.

GEOMETRIC INTERPRETATION



Coordinate transformation from world to camera:

$$\mathbf{X}_C = {}^{C}\mathbf{R}_W \mathbf{X} + {}^{C}\mathbf{t} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

where ${}^{C}\mathbf{t}$ is the world orgin seen from camera.

Rotate and then, translate.

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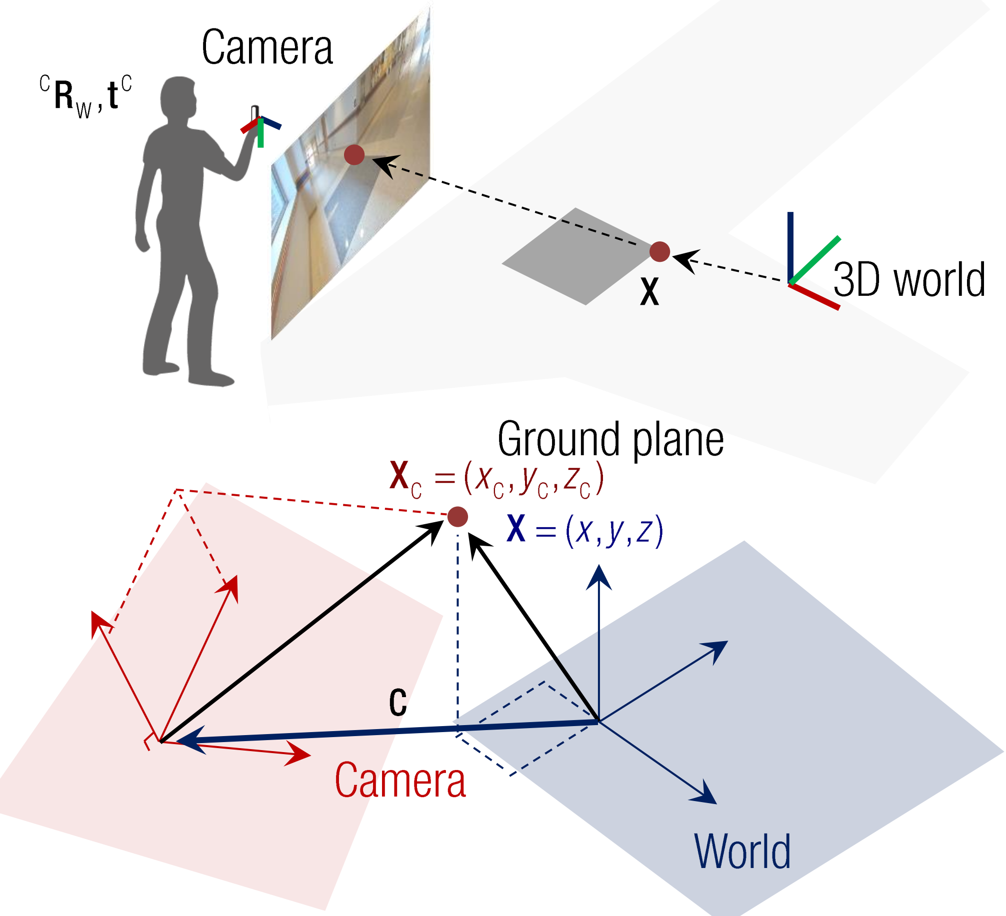
cf) **Translate and then, rotate.**

$$\mathbf{X}_C = {}^{C}\mathbf{R}_W (\mathbf{X} - \mathbf{C}) = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \begin{bmatrix} 1 & & & -C_x \\ & 1 & & -C_y \\ & & 1 & -C_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

where $\mathbf{C}$ is the camera location seen from world.

Camera Projection Matrix



Coordinate transformation from world to camera:

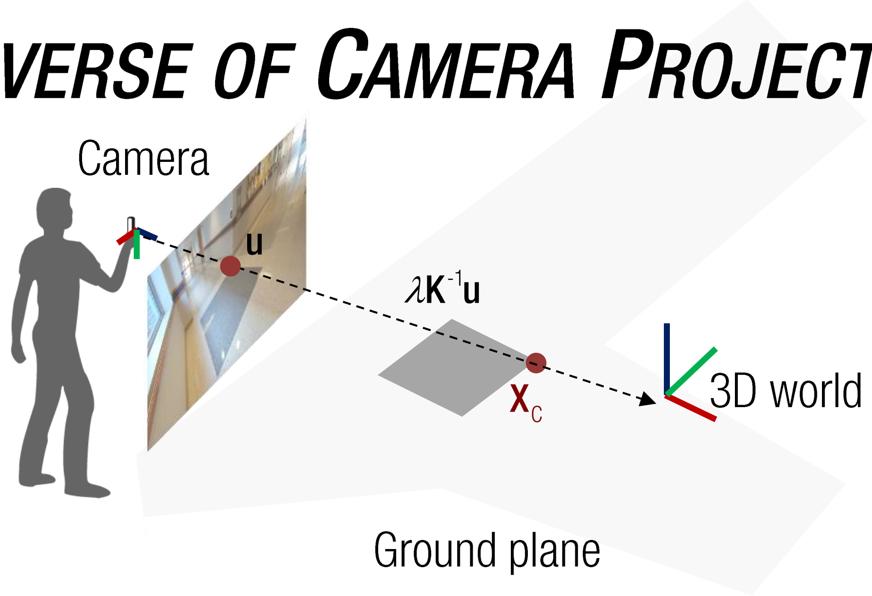
$$\mathbf{X}_C = {}^C\mathbf{R}_W \mathbf{X} + {}^C\mathbf{t} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Camera projection of world point:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

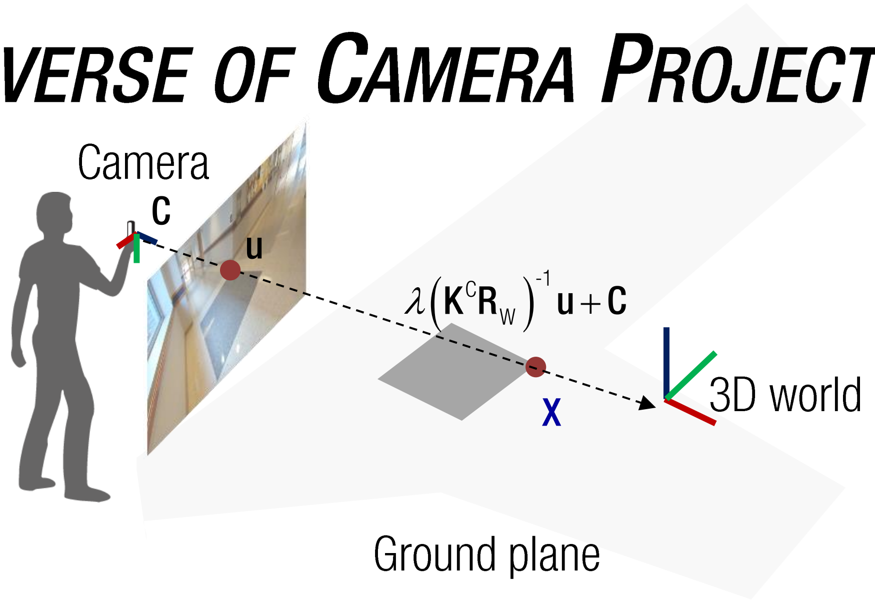
$$= \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \left[\underbrace{\begin{matrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{matrix}}_{{}^C\mathbf{R}_W} \;\; \underbrace{\begin{matrix} t_x \\ t_y \\ t_z \end{matrix}}_{{}^C\mathbf{t}} \right] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

INVERSE OF CAMERA PROJECTION MATRIX



$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{X}_C$$

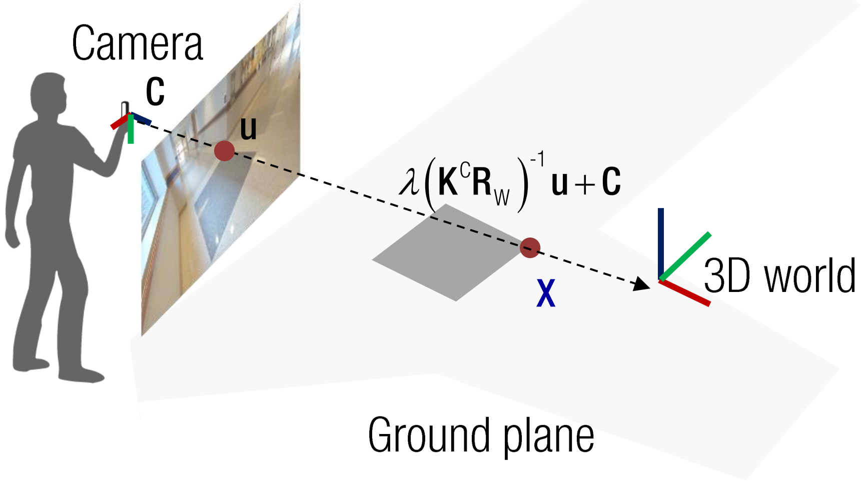
INVERSE OF CAMERA PROJECTION MATRIX



$$\lambda\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{X}_{C}$$

$$= \mathbf{K}^{C}\left(\mathbf{R}_{W}\mathbf{X} + {}^{C}\mathbf{t}\right) = \mathbf{K}\,{}^{C}\mathbf{R}_{W}(\mathbf{X} - \mathbf{C})$$

INVERSE OF CAMERA PROJECTION MATRIX

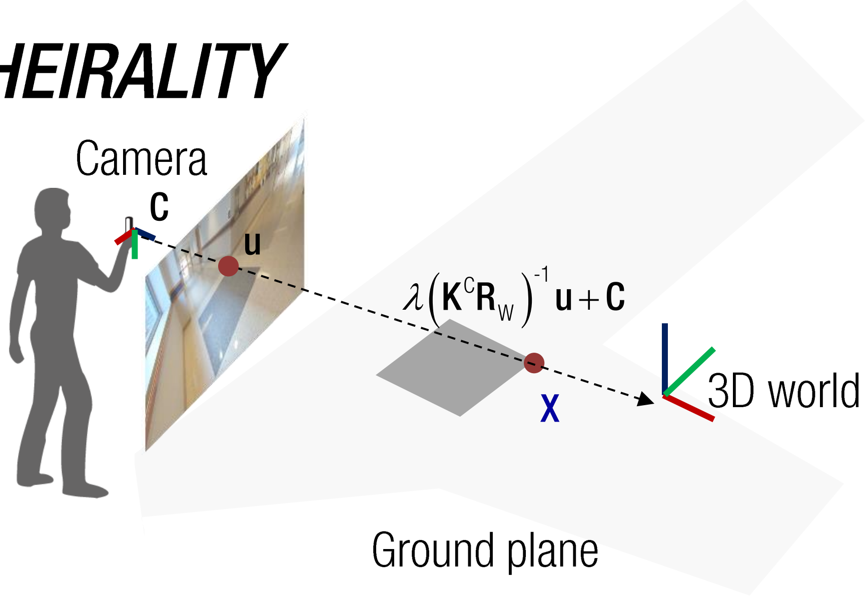


$$\lambda\begin{bmatrix}u\\v\\1\end{bmatrix}=\mathbf{K}\mathbf{X}_{C}$$

$$=\mathbf{K}^{C}\left(\mathbf{R}_{W}\mathbf{X}+{}^{C}\mathbf{t}\right)=\mathbf{K}\,{}^{C}\mathbf{R}_{W}(\mathbf{X}-\mathbf{C})$$

$$\rightarrow\quad \mathbf{X}=\underbrace{\lambda\left(\mathbf{K}\,{}^{C}\mathbf{R}_{W}\right)^{-1}\begin{bmatrix}u\\v\\1\end{bmatrix}}_{\text{3D ray direction}}+\underbrace{\mathbf{C}}_{\text{3D ray origin}}$$

CHEIRALITY



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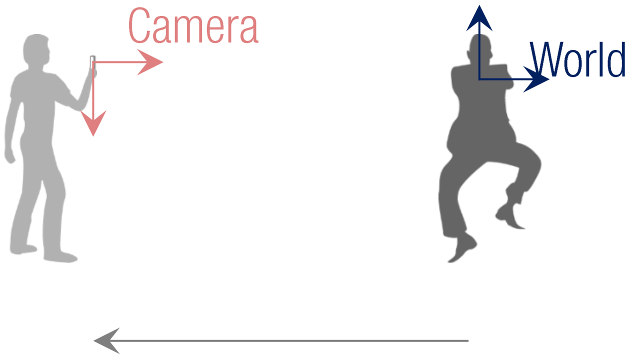
where $\lambda>0$

Full Perspective Model

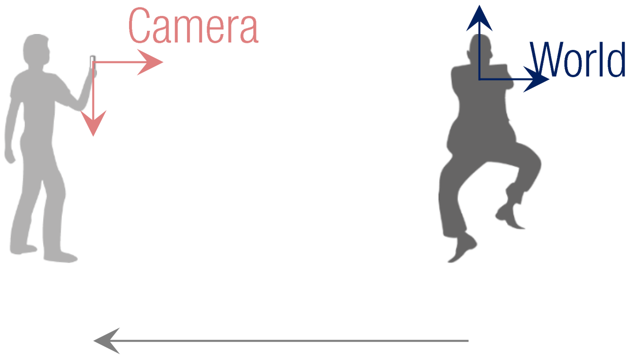


Perspective camera model:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Strong perspectiveness

AFFINE MODEL



Strong perspectiveness

Perspective camera model:

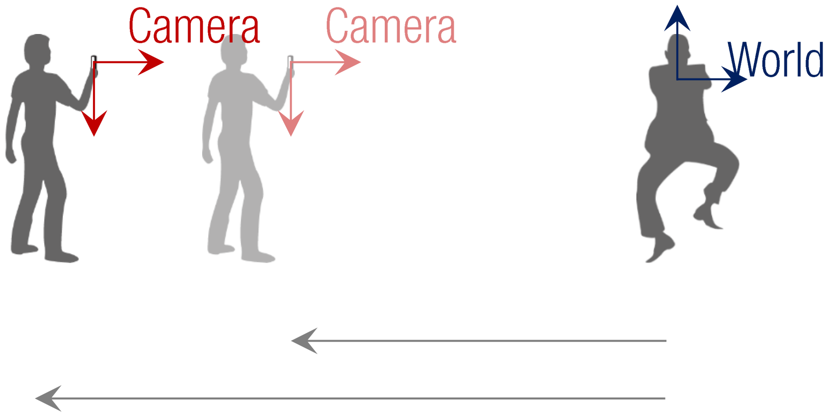
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Affine camera model:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P}_A \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$$\longrightarrow \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{23} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

AFFINE MODEL



Weak perspectiveness

Strong perspectiveness

Perspective camera model:

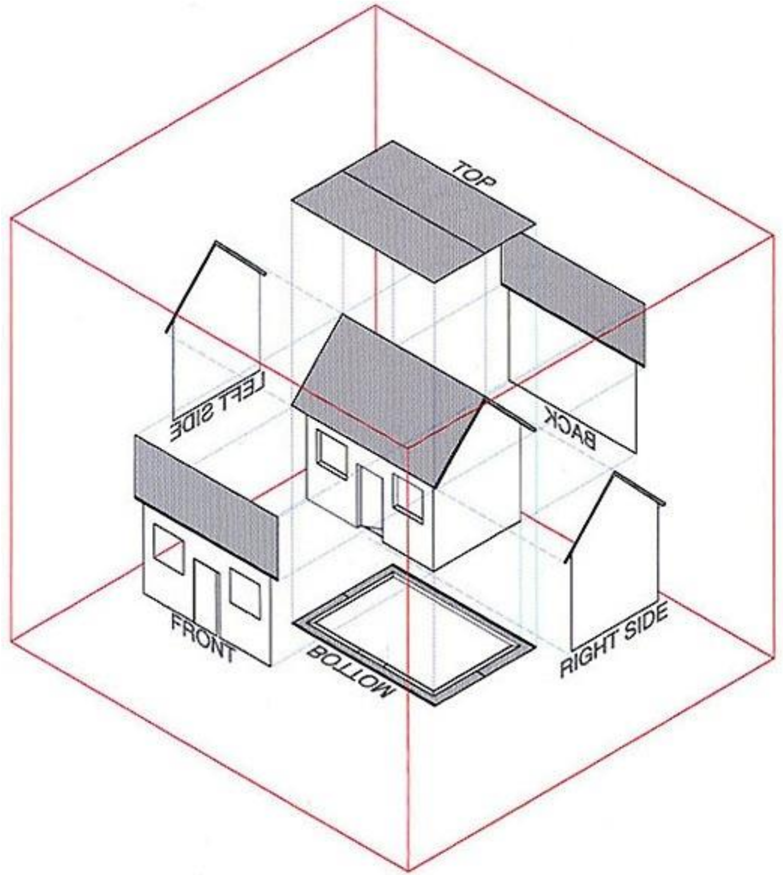
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Affine camera model:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P}_{\mathrm{A}} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$$\longrightarrow \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{23} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

ORTHOGRAPHIC MODEL



Affine camera:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{23} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Orthographic camera:

$f = 1 \quad p_X = p_Y = 0$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

Camera Anatomy

Lens configuration (internal parameter)

$$\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = L\left(\mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} \right)$$

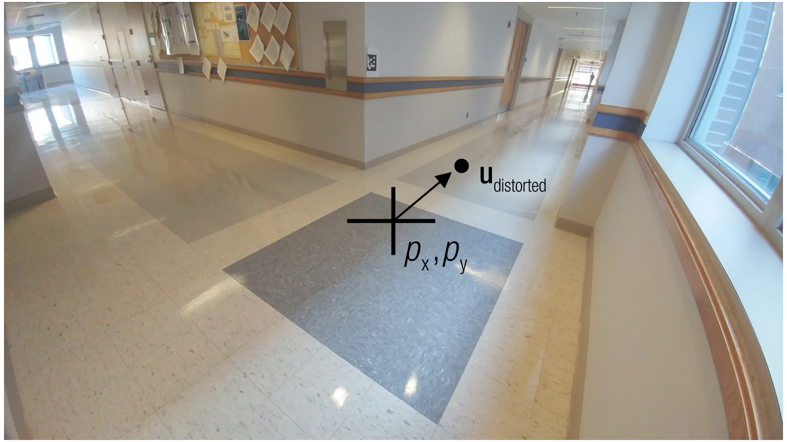
Spatial relationship between sensor and pinhole
(internal parameter)

Camera body configuration
(extrinsic parameter)

Lens Radial Distortion

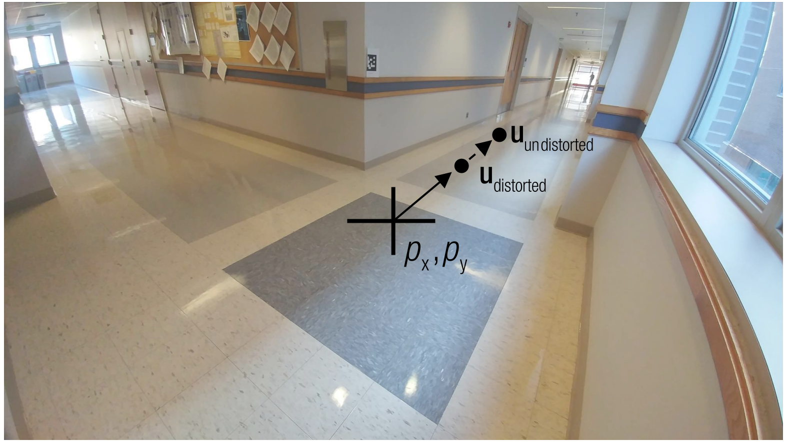
Radial Distortion Model

Assumption: Lens distortion is a function of distance from the principal point.



Radial Distortion Model

Assumption: Lens distortion is a function of distance from the principal point.



$$\bar{\mathbf{u}}_{\text{distorted}} = L(\rho)\bar{\mathbf{u}}_{\text{undistorted}}$$

where $\rho = \left\| \bar{\mathbf{u}}_{\text{distorted}} \right\|$

$$L(\rho) = 1 + k_1\rho^2 + k_2\rho^4 + \cdots$$

Radial Distortion Model

$$\bar{\mathbf{u}}_{\text{distorted}} = L(\rho)\bar{\mathbf{u}}_{\text{undistorted}}$$

$$L(\rho) = 1 + k_1\rho^2 + k_2\rho^4 + \cdots$$

$k_1 < 0$

$k_1 > 0$