

The background features a grey and white checkered floor pattern that recedes into the distance. Overlaid on this are several large, expressive brushstrokes in various colors: a prominent blue stroke at the top, a green stroke to the right, a purple stroke in the center, a red stroke below it, and a thick yellow-green stroke at the bottom. The overall aesthetic is modern and artistic.

# *GOOD FEATURE TO TRACK*

HYUN SOO PARK

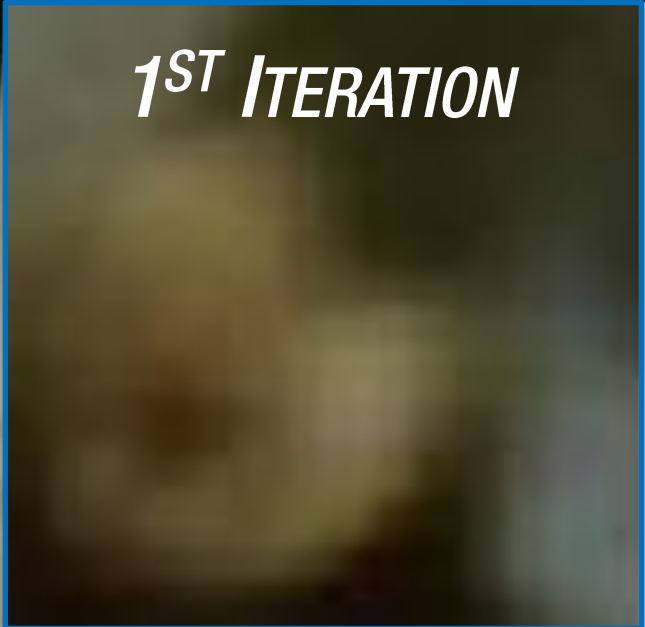
1<sup>ST</sup> ITERATION



$$x = (A^T A)^{-1} A^T b \quad x = \begin{bmatrix} -2.76 \\ 1.27 \end{bmatrix}$$

$I(:, :, t)$

1<sup>ST</sup> ITERATION



$I(:, :, t')$

10<sup>TH</sup> ITERATION



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$I(:, :, t)$

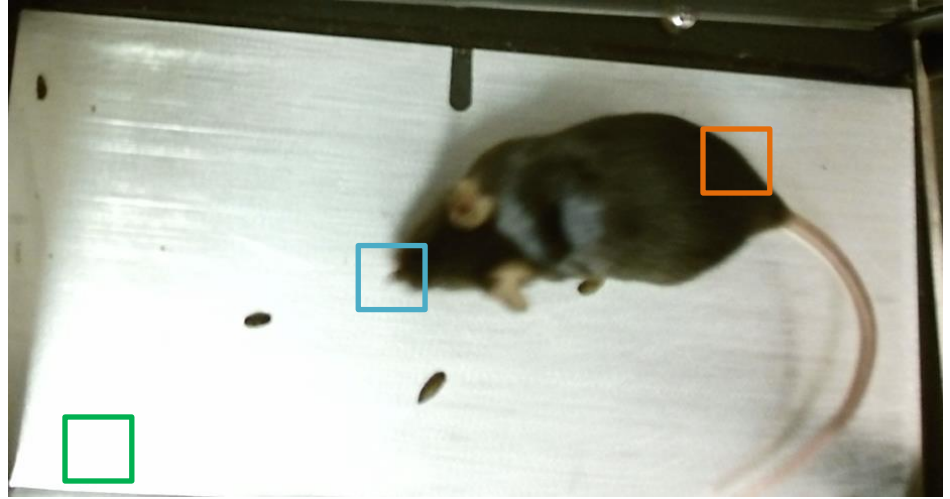
10<sup>TH</sup> ITERATION



$I(:, :, t')$

# *CONDITION NUMBER*

Some patches are better tracked than others.



# SOLVABILITY

$$\underbrace{\begin{bmatrix} I_{x|1} & I_{y|1} \\ \vdots & \vdots \\ I_{x|n} & I_{y|n} \end{bmatrix}}_{n \times 2} \mathbf{A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\text{Unknowns}} = \underbrace{\begin{bmatrix} I_{t|1} \\ \vdots \\ I_{t|n} \end{bmatrix}}_{n \times 1} \mathbf{b}$$

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

Least squares solution

Solvable if the inverse exists:

$$\mathbf{A}^T \mathbf{A} = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

# SOLVABILITY

$$\underbrace{\begin{bmatrix} I_{x|1} & I_{y|1} \\ \vdots & \vdots \\ I_{x|n} & I_{y|n} \end{bmatrix}}_{n \times 2} \mathbf{A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\text{Unknowns}} = \underbrace{\begin{bmatrix} I_{t|1} \\ \vdots \\ I_{t|n} \end{bmatrix}}_{n \times 1} \mathbf{b}$$

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

Least squares solution

Numerical stability  $\sim$  condition number

$$\mathbf{A}^T \mathbf{A} = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$



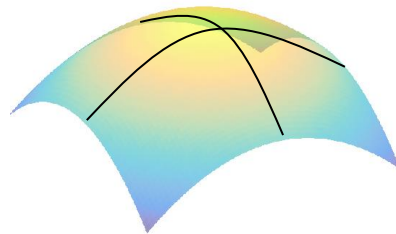
# *CONDITION NUMBER*

$$A^T A = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

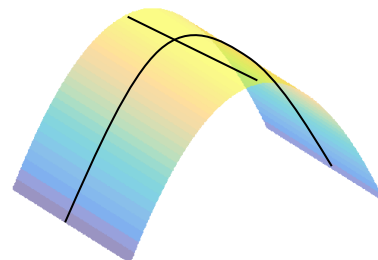
Condition number of matrix:  $\frac{\lambda_1}{\lambda_2}$

where  $\lambda_1, \lambda_2$  are eigenvalues of  $A^T A$   
and  $\lambda_1 \geq \lambda_2$

# RECALL: EDGE THRESHOLDING



$$\lambda_1 \approx \lambda_2$$



$$\lambda_1 \gg \lambda_2$$

Principal curvatures are eigenvalues of Hessian matrix:

$$H = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}$$

# *CONDITION NUMBER*

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$Ax = b$  is well-conditioned if  $\frac{\lambda_1}{\lambda_2} \approx 1$

# CONDITION NUMBER

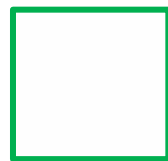
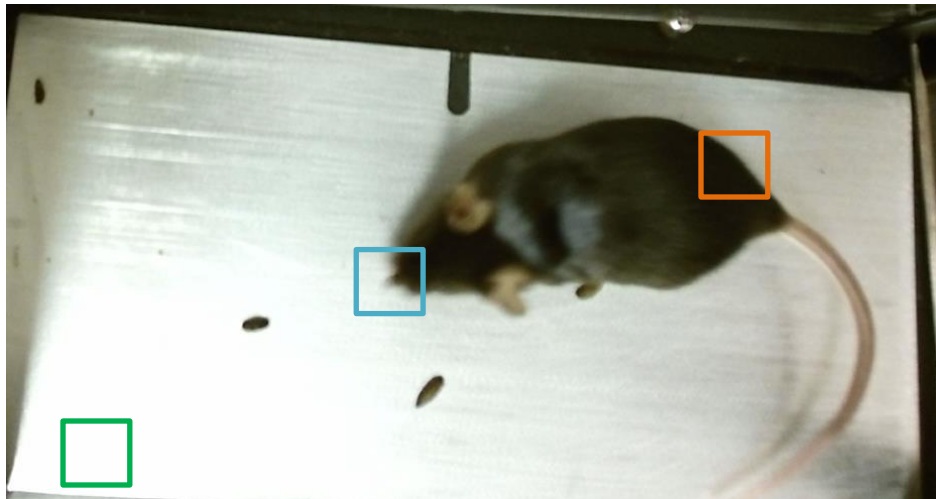
$$A^T A = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

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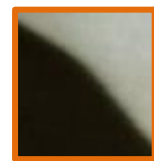
$Ax = b$  is well-conditioned if  $\frac{\lambda_1}{\lambda_2} \approx 1$

Some patches are better tracked than others.



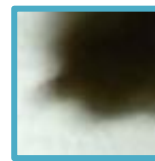
$$\lambda_2 \approx 0$$

$\gg$



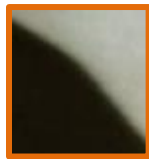
$$\lambda_1 \gg \lambda_2$$

$>$

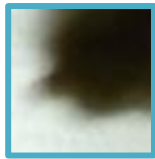


$$\lambda_1 \approx \lambda_2$$

# APERTURE PROBLEM



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$$\lambda_1 \gg \lambda_2$$

$$\lambda_1 \approx \lambda_2$$

There exists an approximate nullspace  $z$ , i.e.,

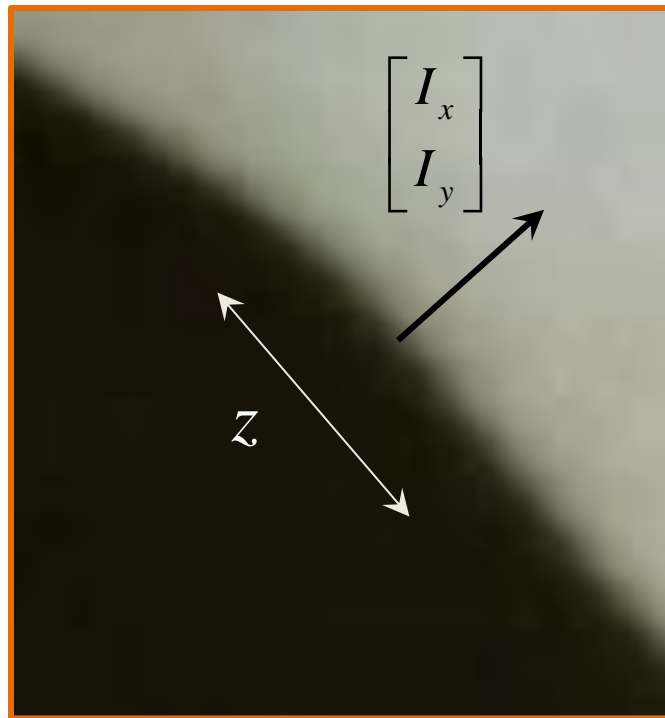
$$Az = 0$$

$$A(x + z) = b$$

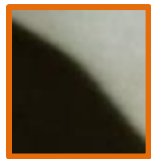
$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} \perp z$$

$z$  is perpendicular to the image gradient.

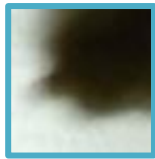
Any motion perpendicular to the dominant image gradient cannot be recovered.



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$$\lambda_1 \gg \lambda_2$$

$$\lambda_1 \approx \lambda_2$$

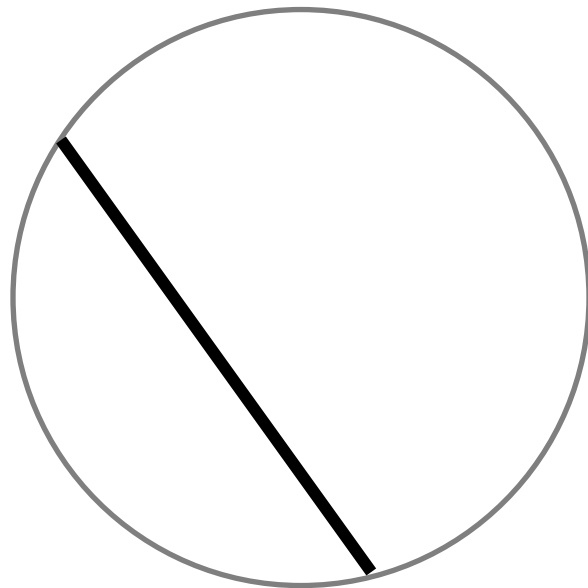
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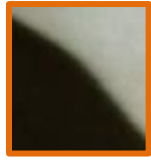
$$A(x + z) = b$$

$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} \perp z$$

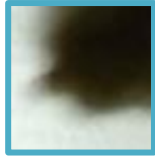
$z$  is perpendicular to the image gradient.



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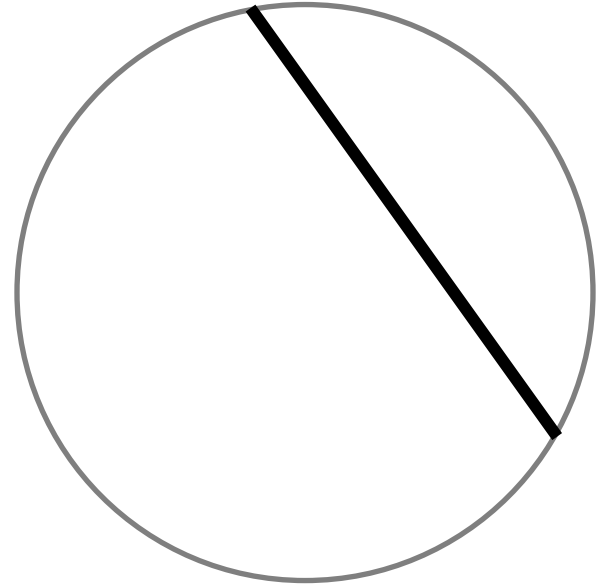
$$Az = 0$$

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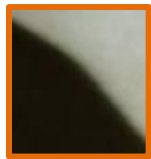
$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} \perp z$$

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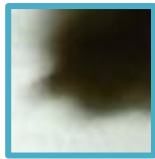
Perceived motion



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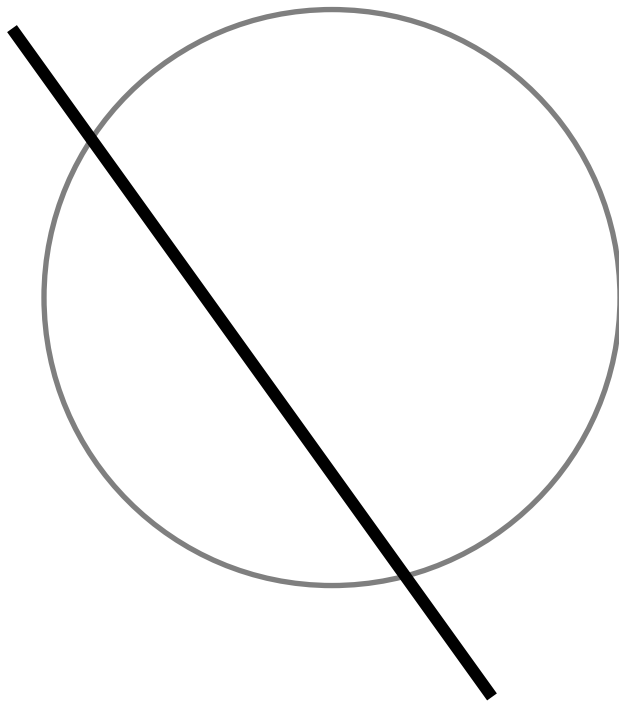
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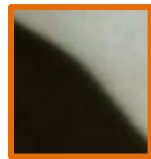
$z$  is perpendicular to the image gradient.

Perceived motion

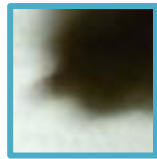




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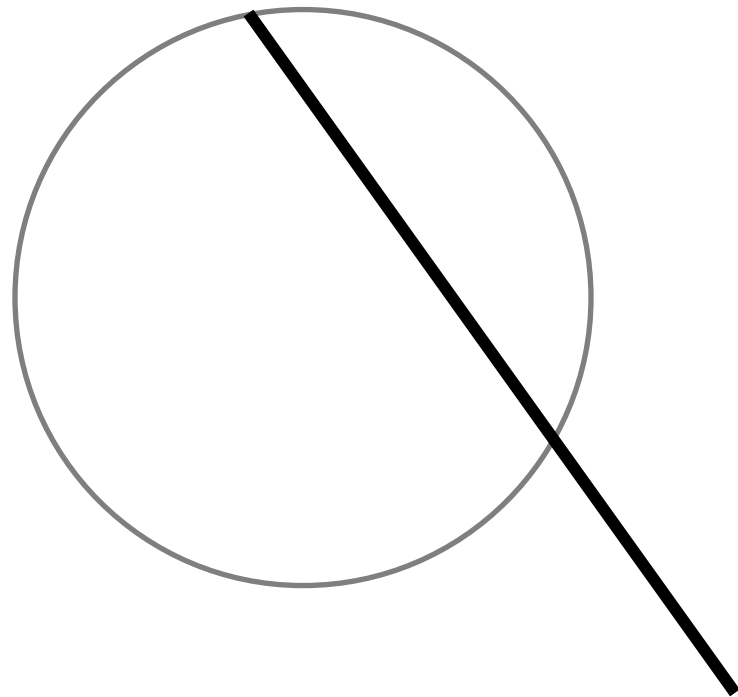
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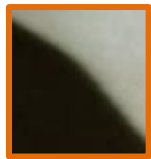
$z$  is perpendicular to the image gradient.

↖  
Perceived motion

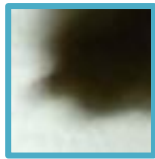
→  
Actual motion



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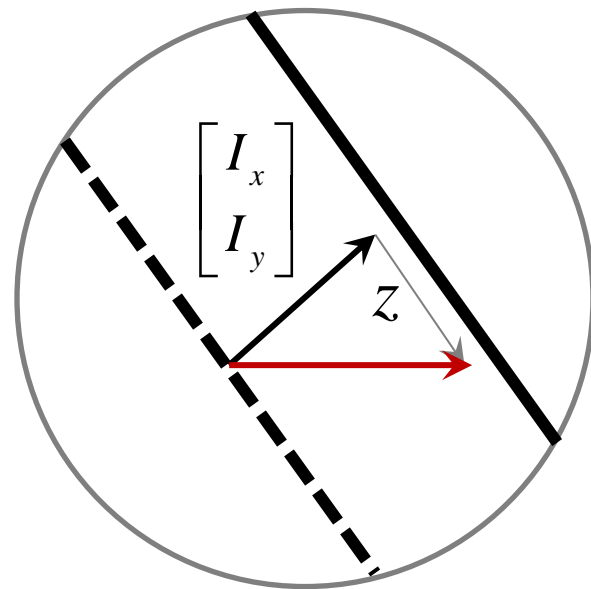
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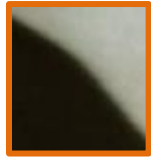
$z$  is perpendicular to the image gradient.

Perceived motion

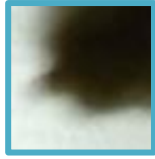
Actual motion



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# GOOD FEATURES TO TRACK



$$\lambda_1 \approx \lambda_2$$

$$A^T A = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

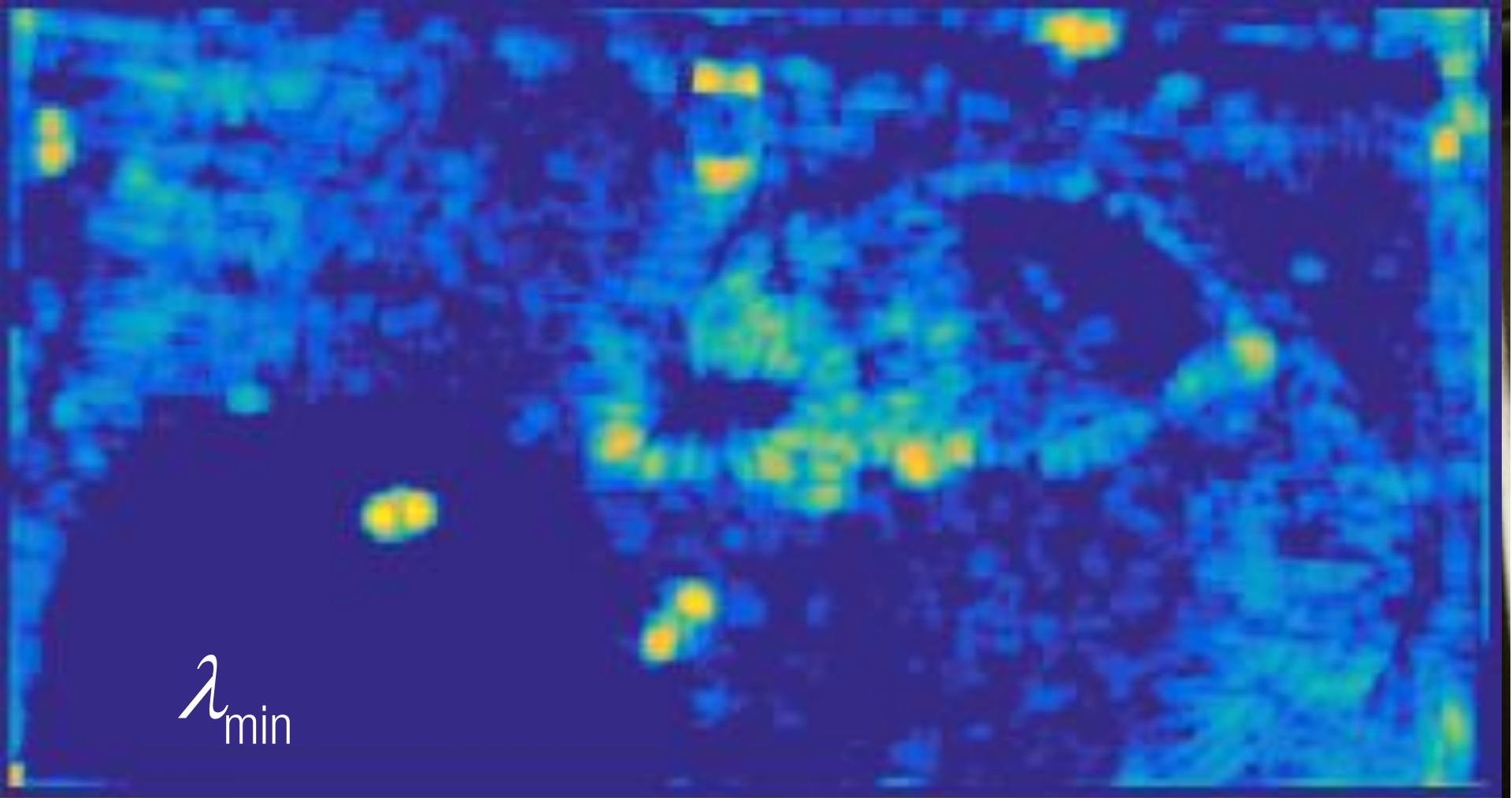
**Shi-Tomasi feature criterion:**

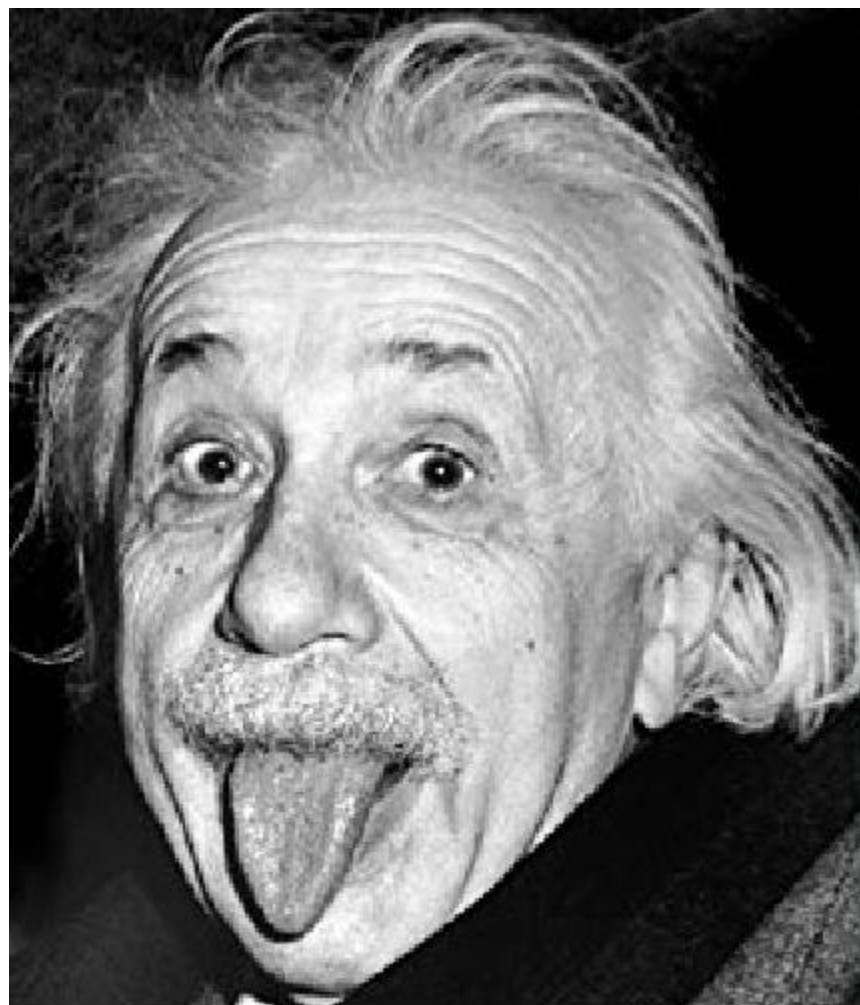
$$\lambda_{\min} > \lambda_{\text{threshold}}$$

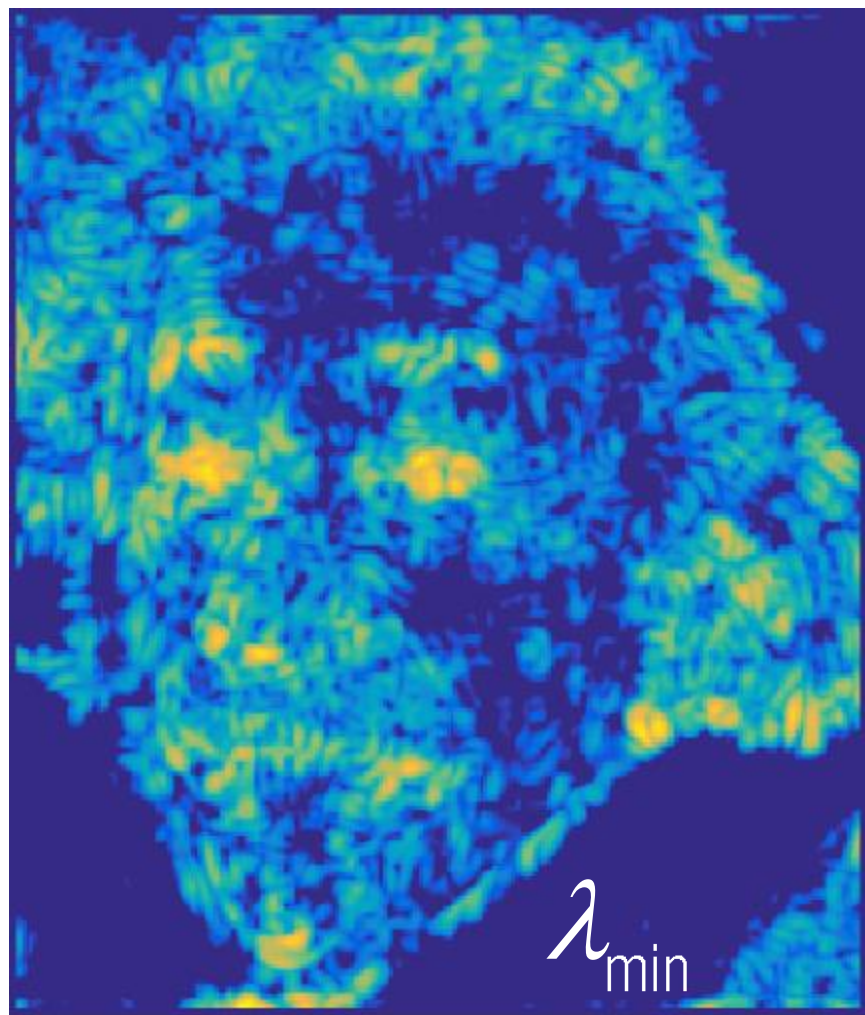
**Harris corner criterion:**

$$\lambda_{\min} \approx \frac{\lambda_{\max} \lambda_{\min}}{(\lambda_{\max} + \lambda_{\min})} = \frac{\det(A^T A)}{\text{trace}(A^T A)} > \lambda_{\text{threshold}}$$

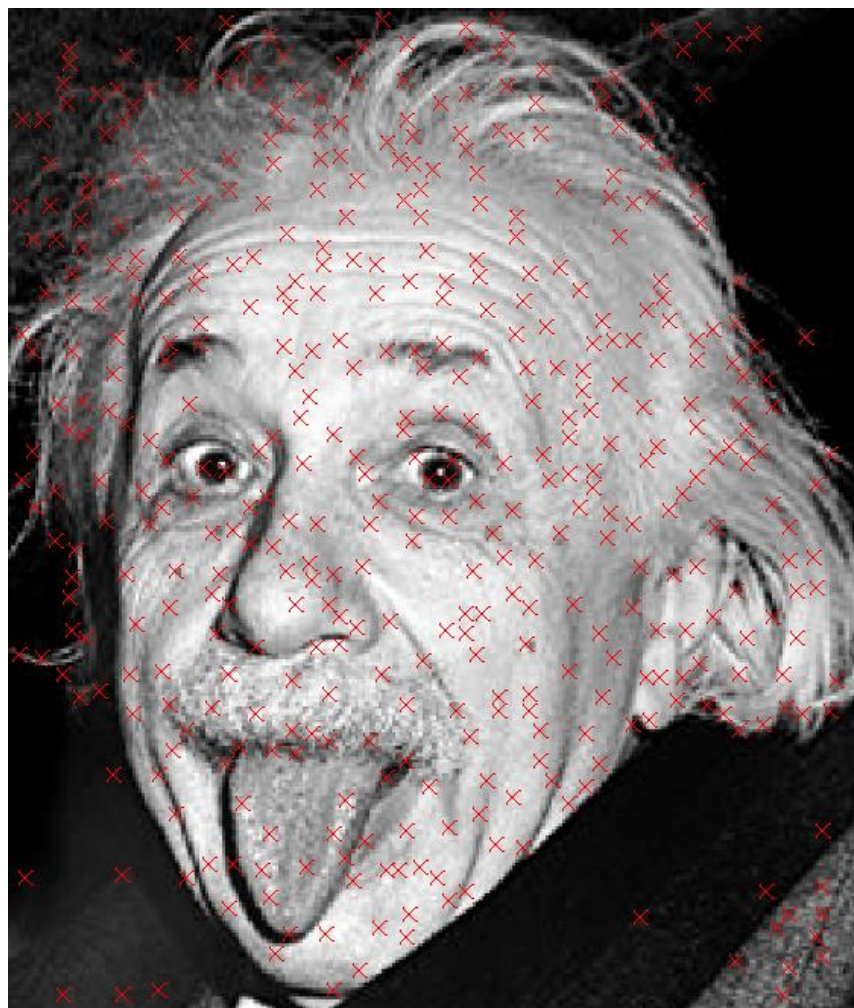












Non-maximum suppression



<https://www.youtube.com/watch?v=JOKTbR-sd6s>