Robot-to-Robot Relative Pose Estimation From Range Measurements
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Abstract—In this paper, we address the problem of determining the 2-D relative pose of pairs of communicating robots from 1) robot-to-robot distance measurements and 2) displacement estimates expressed in each robot’s reference frame. Specifically, we prove that for nonsingular configurations, the minimum number of distance measurements required for determining all six possible solutions for the 3 degree-of-freedom (3-DOF) robot-to-robot transformation is 3. Additionally, we show that given four distance measurements, the maximum number of solutions is 4, while five distance measurements are sufficient for uniquely determining the robot-to-robot transformation. Furthermore, we present an efficient algorithm for computing the unique solution in closed form and describe an iterative least-squares process for improving its accuracy. Finally, we derive necessary and sufficient observability conditions based on Lie derivatives and evaluate the performance of the proposed estimation algorithms both in simulation and via experiments.

Index Terms—Distance measurement, Lie derivatives, observability of nonlinear systems, relative pose estimation.

I. INTRODUCTION AND RELATED WORK

In order to solve distributed estimation problems such as cooperative localization, mapping, and tracking, robots must first determine their relative position and orientation (pose). This extrinsic calibration process is necessary for coordinating a robot team and registering measurements to the same frame of reference. Since the accuracy of the relative (robot-to-robot) transformation can significantly affect the quality of a sensor fusion task (e.g., tracking a target using observations from multiple sensors), it needs to be computed precisely. Mobile robots that move on a plane and use distance and bearing sensors (e.g., laser scanners or stereo cameras) can uniquely determine their relative pose by concurrently processing one distance and two relative bearing measurements recorded in a single location [1]. However, due to power and processing constraints, robots often have to rely on exteroceptive sensors that provide only range measurements. In these cases, computing the unknown robot-to-robot transformation requires developing algorithms for processing multiple distance measurements collected at numerous locations.

Due to their low cost and power consumption, range sensors have recently been widely used for solving localization problems in sensor networks [2] and mobile robots [3]. Range measurements can be obtained using RF or ultrasound signals. Many RF devices are cheap and already integrated into communication equipment; hence, no additional cost is required for range sensing. One of the most common techniques for measuring range is based on the received signal strength indicator (RSSI). The main advantage of RSSI is that it is available in practically all receivers. An example of such a distance-measuring device is the RADAR system employed for wireless network localization in [2]. Alternatively, distances can be measured precisely by estimating the time-of-flight of an acoustic signal transmitted from one sensor or beacon to another [4], [5]. This type of system comprises a transmitter and a receiver. The transmitter simultaneously sends out an RF synchronization message and a distinct sound. The receiver then measures the time difference in the reception of the two signals and computes the distance using a known model for the speed of sound in the air.

Most current research on applications of range sensing has focused on designing algorithms that process distance measurements to determine only the position of each node in a static network of sensors [6], or the position and orientation of a mobile robot when static beacons are deployed within an area of interest [3]. In the case of networks of sensors, a variety of algorithms based on convex optimization [7] and multidimensional scaling (MDS) [8] have been employed to localize the sensor nodes. Additionally, distributed approaches that reduce the communication requirements and better balance the computational load among sensors have also received significant attention in the related literature (e.g., [9] and [10]). In all these cases, the objective is to determine only the position of the sensor nodes with respect to anchor nodes that can globally localize via GPS measurements. Similarly, in the case of mobile robots, the emphasis is on using distance measurements to localize robots with respect to static beacons [3], but not on computing the relative pose of the robots. However, note that determining the robot-to-robot transformation is a prerequisite for efficiently coordinating the motion of teams of robots [11], localizing cooperatively [12], [13], and in general, expressing measurements in a common frame of [1].

The problem we are interested in is that of directly computing the 3 degree-of-freedom (3-DOF) robot-to-robot transformation from distance measurements and displacement estimates, the latter expressed in the reference frame of the corresponding robot. Specifically, we consider pairs1 of communicating robots

1The extension to the problem of multiple robot teams is straightforward once a solution to the pairwise problem is determined.

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equipped with odometric sensors for tracking their motion and range sensors for measuring the distance to each other. In this case, if no prior information about their relative pose is available, a human operator will need to manually measure the transformation between the two robots before they can be deployed to perform their assigned task (e.g., cooperative localization [13]). However, this tedious process limits the accuracy of the robot-to-robot transformation and increases the cost of deploying large teams of robots due to the time and effort required.

A straightforward approach to automating this extrinsic calibration process is for the robots to move randomly, collect distance measurements, and then compute their relative transformation by employing nonlinear least squares. However, any iterative process applied to minimize the nonlinear, in the unknown variables, cost function relies on the existence of an accurate initial estimate in order to converge to the global minimum. Additionally, since the necessary number of range measurements is not known a priori, a conservative strategy would require the robots to spend excessive time and energy measuring their distance numerous times at different locations. Instead, it would be beneficial for the robots to follow a two-step process: 1) Employ a noniterative algorithm to process the minimum number of distance measurements required to compute an initial estimate of their relative transformation, and 2) apply nonlinear least squares to iteratively refine this initial estimate using additional range measurements. This second step can be repeated until the user-specified level of accuracy is reached.

A simplistic method to compute an initial estimate for the 3-DOF transformation in closed form would require the robots to follow a sequence of coordinated motions and measure distances to each other at certain locations and time instants. Specifically, as shown in Fig. 1, if robot $R_2$ remains static while $R_1$ measures its distance to $R_2$ at three different locations (time instants $t_1$, $t_2$, and $t_3$), the position of $R_2$ with respect to $R_1$ can be uniquely determined. In order to compute their relative orientation also, robot $R_2$ will need to move to a new location and remain static again until robot $R_1$ records another three distance measurements (time instants $t_4$, $t_5$, and $t_6$) and triangulates the new relative position of robot $R_2$. Using these two inferred relative position measurements and knowing the direction of motion of $R_2$ (computed from its own odometry), the relative orientation between the two robots can be uniquely determined. The main drawback of this approach is that it requires tight coordination between the robots for performing the sequence of necessary motions and recording the distance measurements at the appropriate locations. Additionally, this initial calibration phase delays the onset of the actual robot task which can be detrimental in time-critical situations involving large robot teams.

In this paper, we address this problem by deriving noniterative algorithms for computing an initial estimate of the 3-DOF robot-to-robot transformation without restricting their motion [14]. Specifically, we prove that when two robots move randomly and collect three distance measurements at different locations, the maximum number of solutions, generically, is 6 (cf. Lemma 1 in Section II). When four range measurements are available, we show that, generically, there exist no more than four solutions (cf. Lemma 2 in Section III). Furthermore, in Section IV (cf. Lemma 3), we prove that for nonsingular configurations, the relative pose of the robots can be uniquely determined given five distance measurements (instead of six based on the simplistic method outlined in Fig. 1). Efficient algorithms for computing all solutions for the cases described before are presented. Additionally, we provide a novel linear algorithm for determining the unique solution (i.e., when five range measurements are available) that minimizes the numerical error in the computed transformation (cf. Section IV-B). In Section V, we describe the nonlinear least-squares algorithm that uses all range measurements available to iteratively refine the initial estimate for the unknown robot-to-robot transformation. In Section VI, we analyze the system observability, and provide necessary and sufficient conditions for observability based on the control inputs applied to the two robots. In Section VII, we present simulation and experimental results that verify the validity of our theoretical analysis. Concluding remarks and future research directions are provided in Section VIII.

II. DETERMINING THE RELATIVE POSE FROM THREE DISTANCE MEASUREMENTS: AT MOST SIX SOLUTIONS

- $\mathbf{p}_j$: Position of frame $\{j\}$, expressed in frame $\{i\}$.
- $\phi_j$: Angle between the $x$-axes of frames $\{i\}$ and $\{j\}$ (i.e., their relative orientation).
- $\mathbf{R}_j$: Rotational matrix that projects vectors expressed in frame $\{j\}$ to frame $\{i\}$.
- $d_{ij}$: Distance between the origins of frames $\{i\}$ and $\{j\}$.
- $\sin(\alpha)$: Short for $\sin(\alpha)$.
- $\cos(\alpha)$: Short for $\cos(\alpha)$.

Consider two robots $R_1$ and $R_2$ whose initial poses (position and orientation) are indicated by the frames of reference $\{1\}$ and $\{2\}$, respectively (cf. Fig. 2). The two robots move randomly through a sequence of odd poses $\{1\}, \{3\}, \ldots, \{2n-1\}$ for $R_1$ and even poses $\{2\}, \{4\}, \ldots, \{2n\}$ for $R_2$ and measure their distance $d_{ij}$, $i \in \{1, 2, \ldots, 2n-1\}$, $j \in \{2, 3, \ldots, 2n\}$ at each of these locations. Without loss of generality, we assume that only one of the robots records range measurements at each location. If both robots measure the same distance, the two measurements can be combined to provide a more accurate estimate of their distance.

Fig. 1. Simplistic approach to determine an initial estimate for the robot-to-robot relative transformation using six distance measurements. The dark (light) triangles denote the locations of robot $R_1$ ($R_2$), and $t_1, t_2, \ldots, t_6 \in \{1, \ldots, 6\}$ indicate the time step(s) that a robot remains at a certain location.
Additionally, the robots are equipped with odometric sensors for estimating their poses with respect to their initial frames of reference, i.e., robot $R_1$ estimates the position vectors $\mathbf{p}_3, \ldots, \mathbf{p}_{2n-1}$ and the orientation angles $\phi_3, \ldots, \phi_{2n-1}$ necessary for determining the rotational matrices $\mathbf{C}_3, \ldots, \mathbf{C}_{2n-1}$. Similarly, the quantities $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and $\phi_1, \ldots, \phi_n$ (and hence, $\mathbf{C}_1, \ldots, \mathbf{C}_n$) are estimated by robot $R_2$ from its own odometry. At this point, we should note that the particular kinematic model used by each robot and the motion measurements available to it do not affect the solution of the relative pose problem. Actually, no motion measurements need to be exchanged between the robots. Instead, only the resulting position displacement estimates (i.e., $\mathbf{p}_2, \ldots, \mathbf{p}_n$) must be communicated.

Our goal is to use the odometry-based estimates and the $n$ distance measurements to determine the maximum number of solutions for the initial 3-DOF robot-to-robot transformation, i.e., their relative position $\mathbf{p}_2$ and orientation $\phi_2 = \phi$, or equivalently (in polar coordinates) $\theta$ and $\phi$, with

$$\mathbf{p}_2 = \rho \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}. \tag{1}$$

Note that $\rho = d_{i2}$ is measured and considered known hereafter.

We first address the case when $n = 3$ distance measurements ($d_{i2}, d_{i4},$ and $d_{i6}$) are available. We proceed by substituting the geometric relations for the position vectors $\mathbf{p}_1, \mathbf{p}_6$ (cf. Fig. 2)

$$\mathbf{p}_4 = \frac{1}{2} \mathbf{C}^T (\mathbf{p}_2 + \frac{1}{2} \mathbf{C}^2 \mathbf{p}_4 - \mathbf{p}_3) \tag{2}$$

$$\mathbf{p}_5 = \frac{1}{2} \mathbf{C}^T (\mathbf{p}_2 + \frac{1}{2} \mathbf{C}^2 \mathbf{p}_6 - \mathbf{p}_3) \tag{3}$$

in the following expressions for the distance measurements $d_{i4}$ and $d_{i6},$ respectively

$$d_{i4}^2 = \frac{3}{5} \mathbf{p}_1^T \mathbf{p}_4 \quad d_{i6}^2 = \frac{1}{5} \mathbf{p}_6^T \mathbf{p}_6. \tag{4}$$

After rearranging terms and substituting $\rho^2 = d_{i4}^2$ for $\mathbf{p}_2^T \mathbf{p}_2,$ these can be written as

$$\begin{aligned}
(\mathbf{p}_2 - \mathbf{p}_3)^T \mathbf{C}^2 \mathbf{p}_4 - \mathbf{p}_2^T \mathbf{p}_3 &= a_0 \\
(\mathbf{p}_2 - \mathbf{p}_3)^T \mathbf{C}^2 \mathbf{p}_6 - \mathbf{p}_2^T \mathbf{p}_5 &= b_0 \tag{5} \\
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= 0.5 (d_{i4}^2 - \rho^2 - 2 \mathbf{p}_4^T \mathbf{p}_4 - \mathbf{p}_3^T \mathbf{p}_3) \\
b_0 &= 0.5 (d_{i6}^2 - \rho^2 - 2 \mathbf{p}_6^T \mathbf{p}_6 - \mathbf{p}_5^T \mathbf{p}_5). \tag{6} \\
\end{aligned}$$

Note that $a_0$ and $b_0$ on the right-hand side of (5) and (6) are known (measured or estimated), while the unknown variables $\theta$ and $\phi,$ embedded in $\mathbf{p}_2$ and $\mathbf{C}$ [cf. (1)], appear only on the left-hand side expressions.

Equations (5) and (6) form a system of two nonlinear equations in the two unknowns $\theta$ and $\phi.$ Applying standard numerical techniques, such as Newton–Raphson [15], for solving this system has three major drawbacks: 1) may take a large number of iterations; 2) requires an accurate initial estimate; and 3) provides no guarantee of finding all solutions. The first drawback is that iterative processes often require a large number of steps before converging to a solution. The second drawback is that in order for the algorithm to converge to the correct answer, initial estimates close to the true values of the unknown variables need to be specified. In practice, however, no such information is available; the only prior knowledge we have for $\theta$ and $\phi$ is that each lies within the interval $[0, 2\pi].$ Finally, if, for example, only $n = 3$ distance measurements are available, the total number of solutions that needs to be determined is six (cf. Lemma 1). To compute all possible roots, numerous initial estimates for the unknowns $\theta$ and $\phi$ must be drawn from the 2-D region $[0, 2\pi] \times [0, 2\pi].$ Such procedure requires a large number of initializations of the iterative process though it provides no guarantees that all six solutions will be computed.\(^3\)

Instead, we hereafter seek to find all possible solutions algebraically. Notice that (5) and (6) can be transformed to polynomial equations by treating $c \phi, s \phi, c \theta,$ and $s \theta$ as four unknown variables. Together with the two trigonometric constraints $s^2 + c^2 = 1$ and $s^2 \theta^2 + c^2 \phi^2 = 1,$ these form a system of four polynomial equations, namely $f_1^\ast, f_2^\ast, f_3^\ast,$ and $f_4^\ast,$ in four unknowns. In general, none of these polynomials lies in the ideal generated by the remaining ones (cf. [16] for the definitions of ideals and varieties), and thus, the square system considered has a finite number of solutions. However, there exist degenerate cases where the system has infinite solutions. For example, when the two robots move on parallel straight lines with equal linear velocities, the bearing angle $\theta$ can take any value within the interval $[0, 2\pi].$ Such a degenerate case occurs because $f_4^\ast$ belongs to further stress this point, consider the case where after numerous initializations of the Newton–Raphson algorithm, only four real roots have been determined. In this situation, there is no clear way to unequivocally declare the remaining two roots complex and terminate the search, or continue the iterations with new initial values in search of the remaining roots.

\(^2\)As will become evident, only the position vectors are required to estimate the unknown robot-to-robot transformation. The orientation angles at intermediate steps are used for expressing the next position vector in the original frame of reference of each robot.
to the ideal \( \langle f_1^*, f_2^*, f_1^*, f_2^* \rangle \), i.e., \( \langle f_1^*, f_2^*, f_1^*, f_2^* \rangle = \langle f_1^*, f_2^*, f_3^* \rangle \).

Based on [16, Prop. 4, Ch. 1, Sec. 4], if \( \langle f_1^*, f_2^*, f_3^*, f_4^* \rangle = \langle f_1^*, f_2^*, f_3^*, f_4^* \rangle \), then the varieties of these two ideals are equal, i.e., \( V(f_1^*, f_2^*, f_3^*, f_4^*) = V(f_1^*, f_2^*, f_3^*) \). Equivalently, the four unknowns are only constrained by three equations, and thus, infinite solutions exist. Note that when the two robots move randomly while satisfying the conditions of Lemma 4 (cf. Section VI-B), the corresponding continuous-time system is locally weakly observable, and hence, within an open neighborhood around a solution of the robot-to-robot transformation, there exist no other points indistinguishable from it (cf. Section VI-B, Definition 4), i.e., generically, the discrete-time system will have a finite number of solutions and the variety \( V(f_1^*, f_2^*, f_3^*, f_4^*) \) has dimension zero. We hereafter concentrate on nonsingular configurations and prove the following lemma.

**Lemma 1**: Given three distance measurements between two robots at three different locations, the maximum number of solutions for the 3-DOF robot-to-robot transformation, generically, is 6.

**Proof**: The following derivation is based on an elimination process for removing the quantities \( c\theta, s\phi, \) and \( c\phi \) from the expressions in (5) and (6), which results in a sixth-order polynomial in the unknown variable \( y := s\theta \). The key idea behind this approach is similar to Gaussian elimination for linear systems of equations. Due to space limitations, only the main steps of this process are shown while reassignment of variables is used to preserve the clarity of presentation.

By substituting the displacement estimates (known from odometry) for the two robots

\[
\begin{align*}
\mathbf{p}_3 &= \begin{bmatrix} a_3 \\ a_2 \end{bmatrix}, & \mathbf{p}_4 &= \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}, & \mathbf{p}_5 &= \begin{bmatrix} b_3 \\ b_2 \end{bmatrix}, & \mathbf{p}_6 &= \begin{bmatrix} b_3 \\ b_4 \end{bmatrix}
\end{align*}
\]

in (5) and (6), and defining as

\[
\begin{align*}
a_5 &:= a_2a_4 + a_1a_3 \\
a_6 &:= a_2a_3 - a_1a_4 \\
b_5 &:= b_2b_4 + b_1b_3 \\
b_6 &:= b_2b_3 - b_1b_4
\end{align*}
\]

we have

\[
\begin{align*}
(\rho a_3c\theta + \rho a_4s\theta - a_5) c\phi + (\rho a_3s\theta - \rho a_4c\theta - a_6) s\phi &= a_6 + \rho (a_1c\theta + a_2s\theta) \\
(\rho b_3c\theta + \rho b_4s\theta - b_5) c\phi + (\rho b_3s\theta - \rho b_4c\theta - b_6) s\phi &= b_6 + \rho (b_1c\theta + b_2s\theta).
\end{align*}
\]

Equations (7) and (8) can be written in a matrix form as

\[
\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} c\phi \\ s\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Note that (9) is linear in the unknowns \( c\phi \) and \( s\phi \). Solving for these two variables, we have

\[
\begin{bmatrix} c\phi \\ s\phi \end{bmatrix} = \frac{1}{\det(u_1w_2 - u_2w_1)} \begin{bmatrix} u_2w_1 - v_1w_2 \\ u_1w_2 - u_2w_1 \end{bmatrix}
\]

where \( \det = u_1w_2 - u_2w_1 \). Substituting these expressions for \( c\phi \) and \( s\phi \) in the trigonometric constraint \( s\phi^2 + c\phi^2 = 1 \) results in a single equation in the variables \( c\theta \) and \( s\theta \)

\[
(u_1v_2 - u_2v_1)^2 = (v_2w_1 - v_1w_2)^2 + (u_1w_2 - u_2w_1)^2
\]

\[
\Rightarrow (u_1v_2 - u_2v_1)^2 = (v_2^2 + v_1^2)w_1^2 + (u_1^2 + u_2^2)w_2^2 - 2(v_1v_2 + u_1u_2)v_1w_1
\]

As detailed in [17], the terms \( v_2^2 + v_1^2, u_2^2, v_1u_2, v_1v_2, u_1u_2, \) and \( u_1v_2 - u_2v_1 \) are all linear in \( c\theta \) and \( s\theta \), while \( v_2^2 + v_1^2, u_2^2, \) and \( u_1v_2 - u_2v_1 \) are all quadratic in the same quantities. Hence, (11) is a third-order polynomial in \( x := c\theta \) and \( y := s\theta \), and can be written in the following simpler form:

\[
f_1 = m_9x^3 + m_8x^2y + m_7xy^2 + m_6x^2 + m_5xy + m_4x
\]

\[
+ m_3y^3 + m_2y^2 + m_1y + m_0 = 0
\]

(12)

and eliminate \( x \) from (12) by using the Sylvester resultant [18]. Specifically, by multiplying (12) with \( x \), and (13) with \( x^2 \), and rewriting the resulting equations in a matrix form, we have

\[
\begin{bmatrix} s_3 & s_2 & s_1 & s_0 & 0 \\ 0 & s_3 & s_2 & s_1 & s_0 \\ 1 & 0 & y^2 - 1 & 0 & 0 \\ 0 & 1 & 0 & y^2 - 1 & 0 \\ 1 & 0 & 1 & 0 & y^2 - 1 \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

with

\[
\begin{align*}
s_0 &:= m_3y^3 + m_2y^2 + m_1y + m_0 \\
s_1 &:= m_7y^3 + m_5y + m_4 \\
s_2 &:= m_9y + m_6, & s_3 &:= m_9.
\end{align*}
\]

For the polynomials in (12) and (13) to have common roots, the determinant of the \( 6 \times 6 \) Sylvester matrix on the left-hand side of (14) must be equal to zero. As shown in [17], the determinant is a sixth-order polynomial in the single variable \( y \)

\[
g_2(y) = \xi_6y^6 + \xi_5y^5 + \xi_4y^4 + \xi_3y^3 + \xi_2y^2 + \xi_1y + \xi_0
\]

(15)

where the constants \( \xi_0, \ldots, \xi_6 \) are functions of the known quantities \( m_0, \ldots, m_9 \). Therefore, the maximum number of solutions for \( y \), including complex roots, is 6.

To prove our claim that there exist at most six solutions for \( \theta \), we also need to show that for every solution of \( y := c\theta \)
[cf. (15)], only one solution for \( x := s \theta \) can be found [cf. (12)]. We present two possible ways for proving this.

By directly computing the Grobner basis [16] for this system of equations, it is easy to show that one base \( g_2 \) is exactly the same as the polynomial in (15), while \( g_1 \) has the form

\[
g_1(x, y) = x + k_5 y^5 + k_4 y^4 + k_3 y^3 + k_2 y^2 + k_1 y + k_0
\]

where the constants \( k_0, \ldots, k_5 \) are functions of known quantities [17]. Therefore, for every root \( y_i^*, i = 1, \ldots, 6 \), of \( g_2(y) = 0 \), there exists only one solution \( x_i^* \) of \( g_1(x, y_i^*) = 0 \) corresponding to it.

Alternatively, we can draw the same conclusion without explicitly computing the Grobner basis. Instead, we only need to show that the leading term of \( g_1 \), \( \text{LT}(g_1) \), is linear in \( x \). This can be easily shown by using the definition of a Grobner basis [16]. A set \( \{g_1, \ldots, g_s\} \subset I \) is a Grobner basis of an ideal \( I \) if the leading term of any element of \( I \) is divisible by one of the \( \text{LT}(g_i) \). Specifically, we can construct one element \( f_3 \) of the ideal \( I = \langle f_1, f_2 \rangle \) by setting

\[
f_3 = f_1 - (m_9 x + m_8 y + m_6) f_2
\]

\[
= (m_7 - m_9) x y^2 + (\text{lower order terms}).
\]

Since the leading term \( \text{LT}(f_3) = (m_7 - m_9) x y^2 \) must be divisible by \( \text{LT}(g_1) \), the degree of \( x \) in \( \text{LT}(g_1) \) has to be 1, or equivalently, \( g_1 \) is linear in \( x \). Hence, the total number of distinct solutions for \( (x, y) \) remains 6.

Finally, \( \phi \) is uniquely determined by back-substitution of \( x = c \theta \) and \( y = s \theta \) in (10).

Although, in general, there exist six, possibly complex, solutions for (15), the total number of real roots will depend on the robots’ trajectories and cannot be determined a priori. A scenario with six real solutions is shown in Fig. 8.

**Corollary 1:** Given three distance measurements between two robots at three different locations, generically, there exist two, four, or six solutions for the 3-DOF robot-to-robot transformation.

This is evident if one considers that \( g_2(y) \) [cf. (15)] is a sixth-order polynomial with real coefficients, and complex roots appear in conjugate pairs.

### A. Computing All Possible Solutions

There exist many methods to compute the roots of a single-variable polynomial. Our approach relies on the eigen-decomposition of the \( 6 \times 6 \) companion matrix [19] for (15).

\[
\begin{bmatrix}
0 & -\xi_0/\xi_5 & -\xi_1/\xi_5 & -\xi_2/\xi_5 & -\xi_3/\xi_5 & -\xi_4/\xi_5 \\
1 & 0 & -\xi_1/\xi_0 & -\xi_2/\xi_0 & -\xi_3/\xi_0 & -\xi_4/\xi_0 \\
0 & 1 & 0 & -\xi_2/\xi_1 & -\xi_3/\xi_1 & -\xi_4/\xi_1 \\
\vdots & & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 & -\xi_5/\xi_4 & -\xi_6/\xi_4
\end{bmatrix}
\]

While this method will determine all six roots of the polynomial (eigenvalues of the companion matrix), only the real ones are of practical interest since they have a geometric interpretation. Once \( y \) is known, \( x \) is determined by computing the null space of the matrix in (14).

### III. Determining the Relative Pose From Four Distance Measurements: At Most Four Solutions

Now, consider the case where the robots \( R_1 \) and \( R_2 \) continue their paths shown in Fig. 2 and move to the new poses \{7\} and \{8\}, respectively, where they record an additional distance measurement \( d_{78} \). We will prove the following.

**Lemma 2:** Given four distance measurements between two robots at four different locations, the maximum number of solutions for the 3-DOF robot-to-robot transformation, generically, is 4.

**Proof:** We proceed in a similar manner as for the case of three distance measurements. Specifically, the new position estimates for the two robots at the locations where they record their fourth distance measurement

\[
[1] p_7 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} [2] p_8 = \begin{bmatrix} e_3 \\ e_4 \end{bmatrix}
\]

are related through the following geometric constraint [analogous to (20)]

\[
[7] p_8 = \frac{1}{2} C^T (1 p_2 + \frac{1}{2} C^2 p_8 - 1 p_7)
\]

Substituting in the expression for the new distance measurement

\[
d_{78}^2 = \frac{1}{2} C^T \cdot \frac{1}{2} C^2 p_8 - 1 p_7^T p_7
\]

is known (i.e., it is computed based on measured and estimated quantities). Following the same algebraic process as in the previous section, (17) can be written as

\[
\begin{align*}
\frac{(\rho e_3 e \theta + \rho e_4 s \theta - e_3) u_1}{u_1} + \frac{(\rho e_3 e \theta + \rho e_4 s \theta - e_6) s \phi}{u_3} = e_0 + \rho (e_1 e \theta + e_2 s \theta)
\end{align*}
\]

where \( e_5 \) and \( e_6 \) are defined as before. Rewriting (7), (8), and (18) in a matrix form, we have

\[
\begin{bmatrix}
\begin{array}{c}
  u_1 \\
  u_2 \\
  u_3
\end{array}
\end{bmatrix}

\begin{bmatrix}
\begin{array}{c}
  v_1 \\
  v_2 \\
  v_3
\end{array}
\end{bmatrix}

\begin{bmatrix}
\begin{array}{c}
  e \phi \\
  s \phi
\end{array}
\end{bmatrix}

= \begin{bmatrix}
\begin{array}{c}
  0 \\
  0 \\
  1
\end{array}
\end{bmatrix}
\]

where the \( u_i, s, v_i \), and \( w_i \), \( i = 1, 2, 3 \), are functions of \( s \theta, e \theta, \) and \( \phi \), known (measured or estimated) quantities. For the earlier system to have nonzero solutions, the determinant of the coefficient matrix must vanish, i.e.,

\[
(u_1 v_2 - u_2 v_1) w_3 + (v_1 u_3 - v_3 u_1) w_2 + (u_2 v_3 - u_3 v_2) w_1 = 0.
\]

Note that the terms \( u_1 v_2 - u_2 v_1, v_1 u_3 - v_3 u_1, \) and \( u_2 v_3 - u_3 v_2 \) are again all linear in \( x := e \theta \) and \( y := s \theta \), and so are \( w_1, w_2, \) and \( w_3 \), which make the previous polynomial quadratic in \( x \) and \( y \) [17]. Following the same elimination procedure as in Section II, we arrive at a fourth-order polynomial in \( y \)

\[
n_1 y^4 + n_3 y^3 + n_2 y^2 + n_1 y + n_0 = 0
\]
where $n_0, \ldots, n_4$ are known constants [17]. In this case, there exist at most four solutions for $y$ all of which can be found in closed form. Once $y$ is determined, back-substitution allows us to find $x$. Finally, $s\phi$ and $c\phi$ are retrieved by computing the null-space vector of the coefficient matrix in (19).

In this lemma, we have shown that the maximum number of possible solutions is 4. We should, however, note that in most cases, in practice, there exists only one real solution for the robot-to-robot transformation. Specifically, even if all four roots of (20) are reals in the interval $[-1,1]$ (and so will be the corresponding values for $x$), back-substitution in (19) will not always give a real value for $\phi$. This is because the first two elements of the null-space vector of the coefficient matrix in (19) must also satisfy the trigonometric constraint $s\phi^2 + c\phi^2 = 1$. Actually, this is a general property of overdetermined nonlinear systems, which we hereafter describe in the context of the problem at hand.

**Corollary 2:** Given four distance measurements between two robots at four different locations, with probability 1 (i.e., almost surely), there exists a unique solution for the 3-DOF robot-to-robot transformation.

To verify this, consider the case where after processing, the first three distance measurements, six solutions are found, denoted as $x_i = [p_2^T \phi_2]^T, i = 1, \ldots, 6$. Without loss of generality, we first assume that $x_1$ corresponds to the true robot-to-robot transformation. When the robots move to their new positions, denoted by frames $\{7\}$ and $\{8\}$, respectively, to record their fourth distance measurement $d_{7,8}$, then $x_1$ should also satisfy (17)

$$
(p_2 - p_7)^T C 2 p_8 - p_2^T p_7 - c_0 = 0 \Leftrightarrow h(x_1, \vartheta) = 0
$$

(21)

where $\vartheta = [p_2^T p_7^T d_{7,8}]^T$. Given $x_1$, we denote by $\mathcal{V}_1$ the set of all values of $\vartheta$ that satisfy (21). Note that $\mathcal{V}_1$ is a 4-D variety.

If we now assume that there exists a second solution, for example, $x_2$, then it should also satisfy (17), i.e.,

$$
h(x_2, \vartheta) = 0.
$$

(22)

As before, given $x_2$, we denote by $\mathcal{V}_2$ the set of all values of $\vartheta$ that satisfy (22).

If both $x_1$ and $x_2$ are valid solutions, then $\vartheta$—which is the realization of the robots’ displacement estimates and the distance measurement—must satisfy (21) and (22), and thus belongs to the set $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$. Note that since $\mathcal{V}$ is constrained by an additional equation [cf. (22)], $\mathcal{V} \subset \mathcal{V}_1$, and the dimension of $\mathcal{V}$ is smaller than that of $\mathcal{V}_1 [16, \text{Th. 3, Ch. 9, Sec. 4}]. Hence, the probability that the robots’ trajectories are such that there exist two solutions is $|\mathcal{V}|/|\mathcal{V}_1|$, which is zero. Following the same process, one can show that the probability of the event that three (or four) solutions exist is also zero.

**IV. Determining the Relative Pose from Five Distance Measurements: Unique Solution**

We now treat the case where the robots $R_1$ and $R_2$ move again and arrive at the locations $\{9\}$ and $\{10\}$, respectively. At that point, they record their fifth distance measurement $d_{9,10}$ and also have the estimates available for their positions $p_9$ and $p_{10}$, respectively. We will first prove that in this case, at most one solution exists (Section IV-A) generically, and then propose an efficient and robust algorithm for computing its value (Section IV-B).

**A. Unique Solution**

**Lemma 3:** Given five distance measurements between two robots at five different locations, generically, there exists at most one solution for the 3-DOF robot-to-robot transformation.

**Proof:** Following the same procedure as in Section II, we arrive at the following four equations (three of these are the ones in (19) and the fourth one is computed in a similar manner using the latest distance measurement):

$$
u_i c \phi + v_i s \phi = u_i, \quad i = 1, \ldots, 4
$$

(23)

where the $u_i$’s, $v_i$’s, and $w_i$’s, are functions of $\theta$, $s \theta$, and known quantities. Choosing all possible pairs of these four equations, we construct $(\frac{1}{2}) = 6$ systems of equations as the ones in (9). Employing the trigonometric constraint $x^2 + y^2 - 1 = 0$ and applying the same elimination process as in Section II, we derive six polynomial equations, each of order 6, in the unknown variable $y := s \theta$ [cf. (15)]

$$
\xi_{0,j} y^6 + \cdots + \xi_{1,j} y + \xi_{0,j} = 0
$$

(24)

where the $\xi_{i,j}$’s, $i = 0, \ldots, 6$, $j = 1, \ldots, 6$, are functions of measured and estimated quantities. Rewriting these polynomials in a matrix form, we have

$$
\begin{bmatrix}
\xi_{6,1} & \cdots & \xi_{1,1} & \xi_{0,1} \\
\vdots & \ddots & \vdots & \vdots \\
\xi_{6,6} & \cdots & \xi_{1,6} & \xi_{0,6}
\end{bmatrix}
\begin{bmatrix}
y^6 \\
y^5 \\
y^4 \\
y^3 \\
y^2 \\
y^1 \\
y^0
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\Leftrightarrow \mathbf{\Xi} \mathbf{y} = \mathbf{0}.
$$

(25)

The null space of matrix $\mathbf{\Xi}$ in (25) has, generically, dimension 1. Thus, a unique solution for the vector $\mathbf{y}$, and hence for $y$, can be determined. Given $y := s \theta$, the unique values of the remaining unknowns $s \theta, c \phi$, and $s \phi$ are computed via back-substitution as in the previous two cases.

Note that there exist singular cases where more than one solution exist and matrix $\mathbf{\Xi}$ will lose rank. For example, when both robots move on straight lines, two symmetric solutions exist (cf. Fig. 3). However, these events will not occur with probability 1 (cf. Corollary 2), and we can conclude that given five distance measurements, generically, there exists at most one solution for the 3-DOF robot-to-robot transformation.

**B. Efficient Computation of the Unique Solution**

The approach for computing the unique solution presented in the previous section requires repetition of the elimination procedure of Section II six times. In addition to being time-consuming, this method may result in incorrect values for the robot-to-robot transformation or even fail due to the accumulation of numerical errors. In this section, we present an alternative
approach based on a linear algorithm for efficiently computing the unique solution given five distance measurements.

Note that this general method for solving systems of polynomial equations has been applied in the past for computing the pose of a camera given four bearing measurements to known landmarks [20], [21]. However, this is the first time that the distance-based robot-to-robot transformation problem is formulated so that it can be amenable to this form of solution.

As described in Sections II, III, and IV-A, for each of the last four distance measurements $d_{134}, \ldots, d_{910}$, we can write an equation similar to (7), repeated as follows after rearranging terms and renaming the known quantities $\alpha_{i,j}$’s:

$$\begin{align*}
\alpha_{7,j} c \phi + \alpha_{6,j} s \phi + \alpha_{5,j} c \theta + \alpha_{4,j} s \theta + \alpha_{3,j} c(\theta - \phi) \\
+ \alpha_{2,j} s(\theta - \phi) + \alpha_{1,j} &= 0, \\
\end{align*}$$

where $j = 1, \ldots, 4$.

The unknowns in these four equations are $c \phi, s \phi, c \theta, s \theta, c(\theta - \phi)$, and $s(\theta - \phi)$. Rewriting them in a matrix form, we have

$$\begin{bmatrix}
\alpha_{7,1} & \ldots & \alpha_{1,1} \\
\vdots & \ddots & \vdots \\
\alpha_{7,4} & \ldots & \alpha_{1,4}
\end{bmatrix}
\begin{bmatrix}
c \phi \\
s \phi \\
c \theta \\
s \theta \\
c(\theta - \phi) \\
s(\theta - \phi) \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \iff Ax = 0$$

where $A$ is the $4 \times 7$ coefficient matrix (known) and $x$ is the unknown vector we want to solve for. Once we have computed the three vectors $r, s,$ and $t$ that span the null space of $A$, $x$ can be written as

$$x = \lambda_1 r + \lambda_2 s + \lambda_3 t$$

(26)

for some scalars $\lambda_1, \lambda_2,$ and $\lambda_3$. To determine their values, we use the trigonometric identities

$$\begin{align*}
c^2 \phi + s^2 \phi &= 1, \\
c^2 \theta + s^2 \theta &= 1, \\
c^2(\theta - \phi) + s^2(\theta - \phi) &= 1, \\
\end{align*}$$

(27)

and the constraint

$$\lambda_1 r_7 + \lambda_2 s_7 + \lambda_3 t_7 = 1$$

(28)

where $r_7, s_7,$ and $t_7$ denote the seventh scalar elements of vectors $r, s,$ and $t$, respectively. Substituting the corresponding elements of $x$ from (26) in the constraints (27) and eliminating $\lambda_3$ using (28), we obtain the following system of equations:

$$\begin{bmatrix}
\beta_{1,1} & \ldots & \beta_{1,5} \\
\vdots & \ddots & \vdots \\
\beta_{5,1} & \ldots & \beta_{5,5}
\end{bmatrix}
\begin{bmatrix}
\lambda_2 \\
\lambda_1 \lambda_2 \\
\lambda_1 \\
\lambda_2 \\
\lambda_2
\end{bmatrix}
= \begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_5
\end{bmatrix}$$

where $\beta_{i,j}$ and $\gamma_i, i, j = 1, \ldots, 5,$ are functions of known quantities [17]. This system can be solved to uniquely determine the unknown vector $[\lambda_1^2 \lambda_2^2 \lambda_1 \lambda_2 \lambda_3]^T$. Given the values of $\lambda_1$ and $\lambda_2$, $\lambda_3$ is computed from (28) and subsequently $x$ is uniquely determined from (26). Finally, the unknown robot-to-robot transformation is retrieved from the first four elements of $x$.

V. WEIGHTED LEAST-SQUARES REFINEMENT

When five or more distance measurements are available to the robots, their relative pose can be computed with higher accuracy based on the following two-step process: 1) Compute the initial estimate for the 3-DOF transformation from five distance measurements (cf. Section IV-B), and 2) use the estimate from the previous step to initialize an iterative weighted least-squares (WLS) algorithm that processes all distance measurements available. We hereafter describe the second step of this process.

Assume that the robots have recorded $n$ distance measurements, which are used to form a system of $n - 1$ nonlinear equations equivalent to (5). Rearranging terms, these can be written in a compact form as

$$h(x, \theta) = 0$$

(29)

where $x := [\theta, \phi]^T$ denotes the vector of the unknown variables and $\theta := [p^T z^T]^T$ is the vector comprising the following known quantities:

$$p := [p_1, \ldots, p_{2n-1}]^T$$

$$z := [d_1, \ldots, d_{2n-1}]^T$$

Since $p, z$, and $p$ are estimated or measured independently, the covariance matrix $P_{\theta \theta}$ of $\theta$ has a block diagonal structure

$$P_{\theta \theta} = \begin{bmatrix}
P_{11} & 0 & 0 \\
0 & P_{22} & 0 \\
0 & 0 & R
\end{bmatrix}$$

(30)

where $P_{11} = E[pp^T]$ and $P_{22} = E[z^2z^T]$, $P_{\theta \theta}$ are the covariance matrices for the robot-position vectors $p$ and $z$.

We denote the error as $\hat{p} = p - p$, where $p$ and $\hat{p}$ are the true and estimated vectors, respectively. Also note that the covariance matrices $P_{11}$ and $P_{22}$ are computed using state augmentation [22]. At every new robot position, a copy of the current robot pose is added to the state vector. The covariance matrix is also appropriately augmented to include the covariance of the duplicate robot pose and its correlations with the rest of the poses in the state vector. Afterward, only the copied state is propagated using odometric measurements. See [23] for details.
respectively, and 

\[ R = \text{diag}(\sigma^2_{d_1}, \ldots, \sigma^2_{d_n}) \]

is the distance measurements’ noise covariance matrix, with \( \sigma_{d_i} \) being its standard deviation.

Given the estimate \( \hat{\theta} \) of \( \theta \), from the robots’ odometry and the recorded distance measurements, and the initial estimate \( \hat{x}^{(1)} \) of \( x \), determined from the algebraic method of Section IV-B, the WLS algorithm computes the new estimate for \( x \) through the following iterative process:

\[ \hat{x}^{(k+1)} = \hat{x}^{(k)} - P_{x\hat{x}}H_x^T(H_{\theta}P_{\theta\theta}H_{\theta}^T)^{-1}h(\hat{x}^{(k)}, \hat{\theta}) \]

where \( P_{x\hat{x}} = [H_x^T(H_{\theta}P_{\theta\theta}H_{\theta}^T)^{-1}H_x]^{-1} \) is the covariance of the estimates, and

\[ H_x = \frac{\partial h}{\partial x} \bigg|_{x = \hat{x}^{(k)}, \theta = \hat{\theta}}, \quad H_{\theta} = \frac{\partial h}{\partial \theta} \bigg|_{x = \hat{x}^{(k)}, \theta = \hat{\theta}} \]

are the Jacobians of the nonlinear function \( h \) [17].

Note that the iterative WLS process will converge if the robot-to-robot transformation is observable. In practice, we can detect singular cases by checking the rank of \( H_x \); if it is not of rank 2, the robots will need to move to new locations and acquire additional range measurements. In order to avoid such singular configurations, in the following section, we present necessary and sufficient conditions on the robots’ motion for the system to be observable.

VI. OBSERVABILITY ANALYSIS

The system describing the 3-DOF robot-to-robot transformation given odometric and distance measurements is nonlinear. Therefore, tests designed for linear time-invariant systems (e.g., the Gramian matrix rank [24] or the Popov–Belevitch–Hautus (PBH) test [25]) cannot be used for examining its observability. Instead, we hereafter employ the observability rank criterion based on Lie derivatives [26] to determine the conditions under which our system is locally weakly observable. Recently, Mariotti et al. [11] have employed this criterion to investigate the observability of 2-D leader–follower formations using only bearing measurements. Martinelli and Siegwart [12] have also used this test to determine only sufficient conditions for the observability of cooperative localization for pairs of mobile robots navigating in two dimensions.

In this paper for the first time, we study the observability of the nonlinear system describing the time evolution of the robot-to-robot transformation given distance measurements, and determine the necessary and sufficient observability conditions on the motion of the two robots. Specifically, after a brief review of the key concepts of observability (Section VI-A), the Lie derivatives, and the observability rank condition (Section VI-B), in Section VI-C, we prove that the sufficient conditions for the robot-to-robot transformation to be locally weakly observable are i)–ii) both robots have nonzero linear velocities and iii) at least one of them has nonzero rotational velocity. Additionally, we prove that conditions i)–ii) are also necessary.

A. Observability of Nonlinear Systems

Consider the state-space representation of the following nonlinear system [27]:

\[
\Sigma \left\{ \begin{array}{l}
\dot{x} = f(x,u), \\
y(t) = h(x)
\end{array} \right. \quad \text{with } y(t) \in \mathbb{R}^m
\]

where \( x \in \mathbb{M} \) (a \( C^\infty \)-connected manifold of dimension \( n \)) is the state vector, \( u = [u_1, \ldots, u_l]^T \in \mathbb{R}^l \) is the vector of control inputs, and \( y = [y_1, \ldots, y_m]^T \in \mathbb{R}^m \) is the measurement vector, with \( y_k = h_k(x), k = 1, \ldots, m \).

**Definition 1:** Two initial states \( x_0 \) and \( x_1 \) are indistinguishable if the given same input \( u(t) \), the system \( \Sigma \) produces the same output \( y(t) \) for both initial states \( x_0 \) and \( x_1 \). The system is termed observable if for all \( x \in \mathbb{M} \), the only state indistinguishable from \( x \) is \( x \) itself.

Notice that observability is a global concept. However, it might be necessary for the state to travel a considerable distance or for a long time to distinguish two points of \( \mathbb{M} \). For this reason, the following local concept is introduced [27].

**Definition 2:** The system \( \Sigma \) is locally observable at \( x_0 \) if for every open neighborhood \( U \) of \( x_0 \), the set of points indistinguishable from \( x_0 \) by trajectories in \( U \) only consists of \( x_0 \) itself. The system \( \Sigma \) is locally observable if it is locally observable for every \( x \in \mathbb{M} \).

In practice, it may be sufficient to distinguish \( x_0 \) only from its neighbors (e.g., when some prior knowledge of \( x_0 \) is available). However, it is possible that \( x_0 \) is indistinguishable from states that are far away. This leads to the concept of weak observability [27].

**Definition 3:** The system \( \Sigma \) is weakly observable at \( x_0 \) if there exists an open neighborhood \( U \) of \( x_0 \) such that the only point in \( U \) that is indistinguishable from \( x_0 \) is \( x_0 \) itself. The system \( \Sigma \) is weakly observable if it is weakly observable at every \( x \in \mathbb{M} \).

Again note that it might be necessary to travel far away from \( U \) to distinguish two points in \( U \). For this reason, the following local concept is introduced [27].

**Definition 4:** The system \( \Sigma \) is locally weakly observable at \( x_0 \) if there exists an open neighborhood \( U \) of \( x_0 \) such that for every open neighborhood \( V \) of \( x_0 \) contained in \( U \), the set of points indistinguishable from \( x_0 \) in \( U \) by trajectories in \( V \) is \( x_0 \) itself. The system \( \Sigma \) is locally weakly observable if it is locally weakly observable for every \( x \in \mathbb{M} \).

The advantage of local weak observability over other concepts is that it has a simple algebraic test that will be described in Section VI-B. Furthermore, and in the context of the robots’ relative pose estimation problem, if the system is locally weakly observable and three or more distance measurements are available, then generically given an initial estimate that is close to the true solution, the iterative algorithm described in Section V (or higher order variants of this) will converge to it. Additionally, if a unique solution exists (cf. Lemma 3, Section IV), the required initial estimate can be computed in closed form (Section IV-B), and the system is locally observable.
B. Lie Derivatives and Nonlinear Observability

We consider the special case of system (30) where the process function $f$ can be separated into a summation of independent functions, each one excited by a different component of the control input vector, i.e.,

$$\begin{align*}
\dot{x} &= f_0(x) + f_1(x)u_1 + \cdots + f_l(x)u_l \\
y &= h(x)
\end{align*}$$

(31)

where $f_0$ is the zero-input function of the process model.

The zeroth-order Lie derivative of any (scalar) function is the function itself, i.e., $\mathcal{L}^0 h_k(x) = h_k(x)$. The first-order Lie derivative of function $h_k(x)$ with respect to $f_i$ is defined as

$$\mathcal{L}_i^1 h_k(x) = \frac{\partial h_k(x)}{\partial x_1} f_i(x) + \cdots + \frac{\partial h_k(x)}{\partial x_n} f_n(x) = \nabla h_k(x) \cdot f_i(x)$$

(32)

where $f_i(x) = [f_{i1}(x), \ldots, f_{in}(x)]^T$, $\nabla$ represents the gradient operator, and “.” denotes the vector inner product. Considering that $\mathcal{L}_i^0 h_k(x)$ is a scalar function itself, the second-order Lie derivative of $h_k(x)$ with respect to $f_i$ is

$$\mathcal{L}_i^2 h_k(x) = \mathcal{L}_i^1 \left( \mathcal{L}_i^1 h_k(x) \right) = \nabla \mathcal{L}_i^1 h_k(x) \cdot f_i(x).$$

(33)

Higher order Lie derivatives are computed similarly. Additionally, it is possible to define mixed Lie derivatives, i.e., with respect to different functions of the process model. For example, the second-order Lie derivative of $h_k$ with respect to $f_j$ and $f_i$, given its first derivative with respect to $f_i$, is

$$\mathcal{L}_{i j}^2 h_k(x) = \mathcal{L}_{i j}^1 \left( \mathcal{L}_i^1 h_k(x) \right) = \nabla \mathcal{L}_{i j}^1 h_k(x) \cdot f_i(x).$$

(34)

Based on the preceding expressions for the Lie derivatives, the observability matrix is defined as the matrix with rows

$$\mathcal{O} \triangleq \{ \nabla \mathcal{L}_{i j}^k h_k(x) | i, j = 0, \ldots, l; k = 1, \ldots, m; \ell \in \mathbb{N} \}.$$  

(35)

The important role of this matrix in the observability analysis of a nonlinear system is captured by [26, Ths. 3.1 and 3.11], repeated next.

Theorem 1 (Observability sufficient condition): If a system satisfies the observability rank condition, then it is locally weakly observable.

Definition 5 (Observability rank condition): The observability rank condition is satisfied when the observability matrix [cf. (35)] is full rank.

Theorem 2 (Observability necessary condition): If a system is locally weakly observable, then the observability rank condition is satisfied generically.

In this case, “generically” means that the observability matrix is full rank everywhere except possibly within a subset of the domain of $x$ [27]. Hence, if the observability matrix is not of sufficient rank for all values of $x$, the system is not locally weakly observable [28].

C. Observability of the 2-D Robot-to-Robot Transformation

In this section, we first derive the continuous-time system model describing the relative pose and orientation of the two robots, and then analyze its observability, when range measurements are available.

Consider two robots that start from initial poses denoted by the (global) frames of reference $\{G_1\}$ and $\{G_2\}$, respectively. After some time, the robots move and their new poses are now depicted in Fig. 4 by the frames $\{R_1\}$ and $\{R_2\}$. The $2 \times 2$ rotational matrices describing vector transformations between these frames satisfy the following relation:

$$G_1^{-1} C(\phi_{G_2}) = G_1^{-1} C(\phi_{R_1}) R_1^{-1} C(\phi) R_2^{-1} C(\phi)$$

or equivalently

$$G_1^{-1} \phi_{G_2} = \phi_{R_1} - \phi_{R_2}$$

(36)

where $\phi$ denotes the relative orientation of the two robots.

Additionally, their positions satisfy the following geometric constraint (cf. Fig. 4):

$$G_1^{-1} p_{G_2} = G_1^{-1} p_{R_1} + G_1^{-1} C(\phi_{R_1}) R_1^{-1} p_{R_2} - G_1^{-1} C(\phi_{G_2}) C(\phi_{G_2}) G_1^{-1} p_{G_2}$$

(37)

where $R_1^{-1} p_{R_2}$ is the robots’ relative position vector.

Differentiating (36) and (37) with respect to time and noting that $G_1^{-1} \dot{\phi}_{G_2} = 0$ and $G_1^{-1} \dot{p}_{G_2} = 0_{2 \times 1}$, we have

$$\dot{p} = -e_1 v_1 + C(\phi)e_1 v_2 - Jp \omega_1$$

(38)

$$\dot{\phi} = -\omega_1 + \omega_2$$

(39)

with

$$p := R_1 p_{R_2} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(40)

where $v_1$ and $\omega_1$, $i = 1, 2$, are, respectively, the linear and rotational velocities of robot $R_i$.

Furthermore, we rearrange the nonlinear kinematic equations [cf. (38) and (39)] in the following convenient form for
computing the Lie derivatives:

\[
\begin{bmatrix}
  p_x \\
  p_y \\
  x \\
  t_1 \\
  t_2 \\
  t_3 \\
  t_4 \\
\end{bmatrix} =
\begin{bmatrix}
  -1 & v_1 + c\phi & v_2 + \frac{p_y}{-p_x} & -1 & \omega_1 & \omega_2 \\
  0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

Finally, in order to preserve the clarity of presentation, the measurement function is chosen to be the squared distance between the two robots divided by two \((d^2/2)\), instead of the distance \(d\), i.e.,

\[
h(x) = \frac{d^2}{2} = \frac{1}{2} \mathbf{P}^T \mathbf{P}.
\]

Note that \(d\) and \(d^2/2\) are both strictly positive, there is a one-to-one correspondence between them, and provide the same information for the spatial relation of the two robots.\(^6\)

We hereafter compute the necessary Lie derivatives of \(h\) and their gradients.

1) Zeroth-order Lie derivative (\(\mathcal{L}^0\)h)

\[
\mathcal{L}^0 h = h = \frac{1}{2} \mathbf{P}^T \mathbf{P}
\]

with gradient

\[
\nabla \mathcal{L}^0 h = [\mathbf{P}^T 0] = [p_x \ p_y 0].
\]

2) First-order Lie derivatives (\(\mathcal{L}^1, \mathcal{L}_f^1\)h)

\[
\mathcal{L}^1 h = \nabla \mathcal{L}^0 h \cdot \mathbf{f}_1 = -p_x
\]

\[
\mathcal{L}^1 h = \nabla \mathcal{L}^0 h \cdot \mathbf{f}_2 = p_x c\phi + p_y s\phi
\]

with gradients

\[
\nabla \mathcal{L}^1 h = [-1 0 0]
\]

\[
\nabla \mathcal{L}^1 h = [c\phi \ \ s\phi \ \ -p_x s\phi + p_y c\phi].
\]

3) Second-order Lie derivatives (\(\mathcal{L}^2, \mathcal{L}^2 f_1, \mathcal{L}^2 f_2\)h)

\[
\mathcal{L}^2 h = (\nabla \mathcal{L}^1 h) \cdot \mathbf{f}_1 = -c\phi
\]

\[
\mathcal{L}^2 h = (\nabla \mathcal{L}^1 h) \cdot \mathbf{f}_2 = -p_y
\]

\[
\mathcal{L}^2 h = (\nabla \mathcal{L}^1 h) \cdot \mathbf{f}_4 = -p_x s\phi + p_y c\phi
\]

with gradients

\[
\nabla \mathcal{L}^2 h = [0 \ 0 \ s\phi]
\]

\[
\nabla \mathcal{L}^2 h = [0 \ -1 \ 0]
\]

\[
\nabla \mathcal{L}^2 h = [-s\phi \ c\phi \ -p_x c\phi + p_y s\phi].
\]

4) Third-order Lie derivatives (\(\mathcal{L}^3, \mathcal{L}^3 f_1, \mathcal{L}^3 f_2\)h)

\[
\mathcal{L}^3 h = (\nabla \mathcal{L}^2 h) \cdot \mathbf{f}_3 = -s\phi
\]

\[
\mathcal{L}^3 h = (\nabla \mathcal{L}^2 h) \cdot \mathbf{f}_4 = s\phi
\]

\[
\mathcal{L}^3 h = (\nabla \mathcal{L}^2 h) \cdot \mathbf{f}_4 = -p_x c\phi - p_y s\phi
\]

with gradients

\[
\nabla \mathcal{L}^3 h = [p_x \ p_y 0]
\]

\[
\nabla \mathcal{L}^3 h = [-s\phi \ c\phi \ -p_x c\phi + p_y s\phi]
\]

\[
\nabla \mathcal{L}^3 h = [0 \ 0 \ s\phi]
\]

\[
\nabla \mathcal{L}^3 h = [0 \ -1 \ 0]
\]

which is also full rank since \(\text{det}(\mathcal{O}^T \mathcal{O}) = 1\). Therefore, in both cases, the observability rank condition is satisfied and the system is locally weakly observable (cf. Theorem 1).

Lemma 5 (Necessary conditions): The system (40), (41) is not locally weakly observable if the following conditions are not satisfied:

1) if \(v_1 \neq 0\).
2) if \(v_2 \neq 0\).

Proof: We hereafter show that when any of these two conditions is violated, the observability matrix is never full rank, and thus (cf. Theorem 2) the system is not locally weakly observable.

1) If \(v_1 = 0\), we can only select Lie derivatives that do not involve \(f_1\). In this case, the observability matrix is

\[
\mathcal{O}_1 = \begin{bmatrix}

\nabla \mathcal{L}^0 h \\
\nabla \mathcal{L}^1 h \\
\nabla \mathcal{L}^2 h \\
\nabla \mathcal{L}^3 h
\end{bmatrix} =
\begin{bmatrix}

p_x & p_y & 0 \\
-c\phi & s\phi & -p_x s\phi + p_y c\phi \\
-s\phi & c\phi & -p_x c\phi + p_y s\phi \\
0 & 0 & s\phi
\end{bmatrix}
\]

which is singular since \(\text{det}(\mathcal{O}_1) = 0\).

2) If \(v_2 = 0\), we can only choose Lie derivatives that do not involve \(f_2\). The observability matrix is now

\[
\mathcal{O}_2 = \begin{bmatrix}

\nabla \mathcal{L}^0 h \\
\nabla \mathcal{L}^1 h \\
\nabla \mathcal{L}^2 h \\
\nabla \mathcal{L}^3 h
\end{bmatrix} =
\begin{bmatrix}

p_x & p_y & 0 \\
-c\phi & s\phi & -p_x s\phi + p_y c\phi \\
-s\phi & c\phi & -p_x c\phi + p_y s\phi \\
0 & 0 & s\phi
\end{bmatrix}
\]

which is also singular.
Hence, it is necessary that both linear velocities \( v_1 \) and \( v_2 \) are nonzero for the robot-to-robot relative pose to be locally weakly observable.

**Corollary 3**: Condition iii) \( \omega_1 \neq 0 \) or \( \omega_2 \neq 0 \) is not necessary for the system (40), (41) to be locally weakly observable.

**Proof**: If \( \omega_1 = \omega_2 = 0 \), then the relative orientation will remain constant, i.e., \( \phi = \phi_0 \), and the system equation becomes

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\phi
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & c\phi_0 \\
p\phi_0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
\tag{42}
\]

which is a special case of (40). The gradients of the nonzero Lie derivatives are

\[
\nabla L_{f_1}^1 h = \begin{bmatrix}
p_x & p_y & 0
\end{bmatrix}
\]

\[
\nabla L_{f_1}^2 h = \begin{bmatrix}
0 & 0 & s\phi_0
\end{bmatrix}
\]

\[
\nabla L_{f_2}^1 h = \begin{bmatrix}
c\phi_0 & s\phi_0 & 0
\end{bmatrix}
\]

\[
\nabla L_{f_2}^2 h = \begin{bmatrix}
c\phi_0 & s\phi_0 & -p_x s\phi_0 + p_y c\phi_0
\end{bmatrix}
\]

The observability matrix is now given by

\[
O_3 = \begin{bmatrix}
-1 & 0 & 0 \\
p_x & p_y & 0 \\
0 & 0 & s\phi_0 \\
c\phi_0 & s\phi_0 & -p_x s\phi_0 + p_y c\phi_0
\end{bmatrix}
\]

which is full rank in general. Therefore, the system is still locally weakly observable (cf. Theorem 1). Hence, iii) \( \omega_1 \neq 0 \) or \( \omega_2 \neq 0 \) is a sufficient but not a necessary condition for the system to be locally weakly observable.

Finally, we note that Lemmas 4 and 5 provide sufficient and necessary conditions on the robots’ motions for the 3-DOF transformation to be \textit{locally weakly} observable. If, additionally, five distance measurements are available (cf. Lemma 3), then generically the system is \textit{locally} observable and the robot-to-robot transformation can be uniquely determined.
Fig. 6. Performance of the WLSs refinement step when using Algorithms IV-A and IV-B for initialization. (a) Final $\theta$ and $\phi$ estimation errors for the cases that converge to the true solution (consistent). (b) Percentage of cases that converge to local minima (inconsistent). (c) Percentage of cases where the maximum number of iterations (1000) is reached. (d) Average number of iterations required to converge.

VII. SIMULATION AND EXPERIMENTAL RESULTS

A. Simulations

The purpose of our simulations is to verify the validity of the presented algorithms, and demonstrate the accuracy and robustness of Algorithm IV-B (cf. Section IV-B) against that of Algorithm IV-A (cf. Section IV-A) for computing the relative pose of two robots using five distance measurements.

For the results shown hereafter, the trajectories and distance measurements were generated as follows: 1) The two robots start at initial positions 10 m apart from each other and record their first distance measurement; 2) each robot moves randomly for approximately 2 m; and 3) the robots record a distance measurement at their new positions. Steps 2) and 3) were repeated until six distance measurements were collected.

Each robot is modeled as a differential two-wheel drive vehicle equipped with encoders measuring the left $v_{li}$ and right $v_{ri}$ wheel velocities, $i = 1, 2$. The linear and rotational velocities of each robot is given by

$$v_i = \frac{v_{li} + v_{ri}}{2}, \quad \omega_i = \frac{v_{ri} - v_{li}}{a}, \quad i = 1, 2$$

where $a = 0.35$ m is the distance between the wheels.

The wheel velocities are commanded to yield constant linear velocity $v_1 = v_2 = 1$ m/s, and rotational velocity $\omega_i$, $i = 1, 2$, which varies between $\pm 2.8$ rad/s following a uniform distribution. The measured wheel velocities are corrupted with zero-mean Gaussian noise with standard deviations $\sigma_{v_{li}} = \kappa v_{li}$ and $\sigma_{v_{ri}} = \kappa v_{ri}$, $i = 1, 2$. The distance measurement noise is also assumed to be additive zero-mean Gaussian with covariance $R = \sigma_d^2 I_n$. Both $\sigma_d$ and $\kappa$ (velocity noise standard deviation as percentage of the actually velocity) were used as parameters in our simulations for examining the robustness and accuracy of the presented algorithms.

Fig. 5 shows the averaged results of 2000 trials for each noise level. In particular, Fig. 5(a) and (c) shows the relative bearing
\( \theta \) error as a function of the noise standard deviation in the odometry and the distance measurements. The corresponding figures for the relative orientation \( \phi \) error are similar to the ones for bearing and are omitted due to space limitations. To further highlight the difference in performance of the two algorithms, we fixed the standard deviation of the noise in the distance (odometry) measurements, and plotted the estimation error as a function of the odometry (distance) noise standard deviation in Fig. 5(b) [Fig. 5(d)]. Evidently, the linear Algorithm IV-B is consistently more accurate than Algorithm IV-A over a wide range of values for \( \kappa \) and \( \sigma_d \).

As explained in Section V, in order to improve the accuracy of the relative pose estimates, a WLS refinement step is also necessary. However, every iterative method requires an initial estimate whose accuracy greatly impacts the quality of the solution. Imprecise initial estimates can lead to local minima and even divergence. Fig. 6 compares the WLS performance when using the resulting estimates from Algorithms IV-A and IV-B for initialization. Fig. 6(a) shows the estimation errors when the WLS process converged to the true solution and the estimates are consistent. As evident, the estimation errors are comparable regardless of which algorithm is used for initialization. However, the impact of the initialization algorithm on the WLS’s performance is better appreciated when considering additional factors such as 1) percentage of inconsistent estimates, 2) percentage of divergence, and 3) required number of iterations. Specifically, the percentage of inconsistent estimates \( 7 \) is significantly larger when Algorithm IV-A, instead of IV-B, is used for WLS initialization [cf. Fig. 6(b)]. Furthermore, as shown in Fig. 6(c) and (d), initialization with Algorithm IV-B reduces the number of WLS iterations required to converge and improves the divergence rate (we declare that the WLS process has not converged if it has reached 1000 iterations). Finally, we note that the estimation accuracy is significantly improved when additional distance measurements are available [cf. Fig 6(a)].

**B. Experiments**

We hereafter experimentally validate the accuracy of the algorithms presented in Sections II–IV for determining the robot-to-robot transformation given three, four, or five range measurements.

For our experiments, we deployed two Pioneer II robots within an area of \( 4 \text{ m} \times 5 \text{ m} \) (cf. Fig. 7). The robots estimated their poses with respect to their initial locations using linear and rotational velocities measurements from their wheel encoders. An overhead camera was used to provide ground truth for evaluating the errors in the computed estimates. Additionally, using the position measurements from the camera, we were able to compute the distances between the robots and control their accuracy by adding noise in these measurements. In this experiment, the standard deviation of the noise in the distance measurements was set to \( \sigma_d = 0.01 \text{ m} \).

Given three distance measurements, the six possible robot-to-robot transformations are shown in Fig. 8. By comparing each of them to the true robot trajectory (cf. Fig. 7), it is evident that solution (d) [cf. Fig. 8(d)] corresponds to the true configuration. The same conclusion can be reached after processing four or five distance measurements (in this case, there exists only one real solution when considering four distance measurements).

The computed values for the two robot’s relative bearing and orientation are shown in the first three columns of Table I for three, four, and five distance measurements, respectively. Note that for the case of three distance measurements, only the solution closest to the true value (shown in the last column and computed using the camera) is included. These data show a slight decrease in accuracy when using additional distance measurements. This is to be expected since the distance measurements are used here to compute the initial estimate, at time \( t = 0 \), for the robot-to-robot transformation, i.e., when the two robots first detected each other. In this case, as the robots move and accumulate odometry error, the uncertainty in the initial transformation will increase. However, this would not be, in general, the case if instead we were to compute the robot-to-robot transformation at subsequent time instants. Actually, as it was shown in Section VI that the system describing the current robot-to-robot transformation is locally observable, and hence, its error will remain bounded.

Finally, we tested the accuracy of the WLS process initialized with the estimate computed by Algorithm IV-B given five distance measurements. As shown in the fourth column of Table I, the WLS estimates for the relative bearing and orientation are \( \theta = -0.9319 \text{ rad} \) and \( \phi = 0.3328 \text{ rad} \), both of which are of higher accuracy compared with the initial estimates.

\( 7 \)An estimate is considered inconsistent if the estimation error is larger than \( 3\sigma \), where \( \sigma \) is obtained from the diagonal elements of the covariance matrix \( P_{xx} \) computed using the iterative WLS algorithm (cf. Section V).

![Fig. 7. Trajectories of the two robots and the locations where distance measurements were recorded.](image-url)
TABLE I
RESULTS WITH THREE, FOUR, AND FIVE DISTANCE MEASUREMENTS

<table>
<thead>
<tr>
<th>No. meas</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>5 (WLS)</th>
<th>Cam</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ (rad)</td>
<td>-0.9490</td>
<td>-0.9469</td>
<td>-0.9501</td>
<td>-0.9319</td>
<td>-0.9160</td>
</tr>
<tr>
<td>$\phi$ (rad)</td>
<td>0.3312</td>
<td>0.3286</td>
<td>0.3353</td>
<td>0.3328</td>
<td>0.3280</td>
</tr>
</tbody>
</table>

VIII. CONCLUSION AND FUTURE WORK

In this paper, we presented efficient algorithms to solve the relative pose problem for pairs of robots moving on a plane using robot-to-robot distance measurements. Noniterative algorithms for computing the initial estimate of the 3-DOF transformation were presented for the cases when three, four, and five distance measurements were available. We have shown that for nonsingular configurations, the maximum number of solutions for the aforementioned cases are six, four, and one, respectively. In addition, we presented a novel linear algorithm for computing the unique solution that is robust to numerical errors. We have shown that for nonsingular configurations, the maximum number of solutions for the aforementioned cases are six, four, and one, respectively. In addition, we presented a novel linear algorithm for computing the unique solution that is robust to numerical errors. A key advantage of our approach is that it does not require any robot coordination or specific motion strategies, thus reducing the time and effort required. Furthermore, a WLS process was presented that uses the result of the noniterative algorithm as the initial relative pose estimate and iteratively improves its accuracy. Finally, we presented necessary and sufficient observability conditions for the motion of the robots based on Lie derivatives.

Currently, we investigate optimal motion strategies for minimizing the uncertainty in the relative pose estimates and seek to extend the results of this paper to motion in three dimensions.

REFERENCES


