Visual Odometry with Directional Constraints

Xun Zhou and Stergios I. Roumeliotis
{zhou|stergios}@cs.umn.edu

Department of Computer Science & Engineering
University of Minnesota

MARS LAB

Multiple Autonomous Robotic Systems Laboratory

Technical Report
Number -2010-002
Feb 2010

Dept. of Computer Science & Engineering
University of Minnesota
4-192 EE/CS Building
200 Union St. S.E.
Minneapolis, MN 55455
Tel: (612) 625-2217
Fax: (612) 625-0572
URL: http://www.cs.umn.edu/~zhou
1 Problem Description

Given three image point correspondences $p_i$ and $p'_i$, $i = 1, \ldots, 3$, and bearing measurements $b$ and $b'$ towards a reference direction in two calibrated views, we want to determine the rotation matrix $R$ and the up-to-scale translation vector $t$ between these two views.

The pair of bearing measurements provide us the constraint

$$b = Rb'$$

which allows us to determine the rotation matrix $R$ up to two degrees of freedom. Only the rotation around $b$ is unknown. Without loss of generality, we assume the undetermined rotation is around $e_1 = [1, 0, 0]^T$, hence the rotation matrix is of the form

$$R(e_1, \alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & c & -s \\
0 & s & c
\end{bmatrix}, \quad c = \cos(\alpha), \quad s = \sin(\alpha).$$

Additionally, we assume the translation is of the form $t = [x, y, 1]^T$. Hence, we need to solve 4 unknown parameters ($x$, $y$, $c$, $s$). From the three pairs of image point correspondences, we have three epipolar constraints.

$$p_i^T [t \times] R(e_1, \alpha) p'_i = 0$$

where $p_i = [p_{ix}, p_{iy}, 1]^T$, and $p'_i = [p'_{ix}, p'_{iy}, 1]^T$. Expanding the above equations, we have three bilinear polynomial equations in the four unknowns.

$$a_{i1}cx + a_{i2}sx + a_{i3}cy + a_{i4}sy - a_{i4}c + a_{i3}s + a_{i5}y + a_{i6} = 0, \quad i = 1, \ldots, 3$$

where

$$a_{i1} = p'_{iy} - p_{iy}$$
$$a_{i2} = -p'_{iy}p_{ix} - 1$$
$$a_{i3} = p_{ix}$$
$$a_{i4} = p'_{iy}p_{ix}$$
$$a_{i5} = -p'_{ix}$$
$$a_{i6} = p'_{ix}p_{iy}.$$

And we also have the trigonometric constraint

$$c^2 + s^2 = 1.$$
2 Solution

We will show in this section that the system has four solutions. We will solve this system by elimination and back-substitution. Hereafter we will show an elimination procedure to obtain a univariate polynomial in s. Then we can find the solutions for s from this univariate polynomial, and by back-substitution, we can solve the other three variables. In the back-substitution steps, each solution of s returns one solution for the other three variables, therefore, we have a total of 4 solutions for the relative rotation matrix and translation vector.

The main steps of the elimination procedure are list as follows.

1. Solve for x and y as a function of c and s using the first two equations in (4). x and y can be expressed as quadratic functions of c and s.

2. Substitute x and y in the third equation in (4). This is again a quadratic polynomial in c and s, since there are constraints between the coefficients of cy and s, and sy and c [see equation (4)].

3. Finally, using the Sylvester resultant we can eliminate one of the remaining two unknowns, say c, and obtain a 4th order polynomial in s.

Now, we describe the details of our approach. Rewrite the first two equations in (4) as linear functions of c and s.

\[
\begin{bmatrix}
  a_{11}c + a_{12}s \\
  a_{21}c + a_{22}s
\end{bmatrix}
= \frac{1}{\text{det}}
\begin{bmatrix}
  a_{14}c - a_{13}s - a_{16} \\
  a_{24}c - a_{23}s - a_{26}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Then we solve for x and y from the above linear system.

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \frac{1}{\text{det}}
\begin{bmatrix}
a_{23}c + a_{24}s + a_{25} \\
-(a_{13}c + a_{14}s + a_{15})
\end{bmatrix}
\begin{bmatrix}
a_{14}c - a_{13}s - a_{16} \\
a_{24}c - a_{23}s - a_{26}
\end{bmatrix}
\]

where

\[
\text{det} = (a_{11}c + a_{12}s)(a_{23}c + a_{24}s + a_{25}) - (a_{21}c + a_{22})(a_{13}c + a_{14}s + a_{15}).
\]

Substituting the expression for x and y and s in the third equation in (4) and multiplying both sides of the equation with \(\text{det}\), we get a quadratic polynomial in c and s.

\[
\begin{align*}
a_{31}cx + a_{32}sx + a_{33}cy + a_{34}sy - a_{34}c + a_{33}s + a_{35}y + a_{36} &= 0 \\
\Rightarrow g_1c^3 + g_2c^2s + c_1s^2 + g_3c^2 + g_4cs + g_5s^2 + g_6c + g_7s &= 0 \\
\Rightarrow g_3c^2 + g_4cs + g_5s^2 + (g_1 + g_6)c + (g_2 + g_7)s &= 0
\end{align*}
\]

where

\[
\begin{align*}
g_1 &= a_{14}a_{23}a_{31} - a_{11}a_{23}a_{34} + a_{21}a_{13}a_{34} + a_{11}a_{24}a_{33} - a_{21}a_{14}a_{33} - a_{24}a_{13}a_{31} \\
g_2 &= a_{14}a_{23}a_{32} - a_{21}a_{13}a_{32} + a_{12}a_{24}a_{33} - a_{14}a_{22}a_{33} - a_{12}a_{23}a_{34} + a_{22}a_{13}a_{34} \\
g_3 &= -a_{23}a_{16}a_{31} + a_{25}a_{14}a_{31} + a_{13}a_{26}a_{31} - a_{15}a_{24}a_{31} - a_{11}a_{26}a_{33} + a_{21}a_{16}a_{33} + a_{11}a_{24}a_{35} \\
- a_{21}a_{14}a_{35} - a_{11}a_{25}a_{34} + a_{15}a_{21}a_{34} + a_{11}a_{23}a_{36} - a_{21}a_{13}a_{36} \\
g_4 &= -a_{23}a_{16}a_{32} - a_{24}a_{16}a_{31} + a_{25}a_{14}a_{32} - a_{25}a_{13}a_{31} + a_{13}a_{26}a_{32} + a_{14}a_{26}a_{31} - a_{15}a_{24}a_{32} \\
+ a_{15}a_{23}a_{31} - a_{11}a_{26}a_{34} - a_{12}a_{26}a_{33} + a_{21}a_{16}a_{34} + a_{22}a_{16}a_{33} - a_{11}a_{23}a_{35} + a_{12}a_{24}a_{35} \\
+ a_{21}a_{13}a_{35} - a_{14}a_{22}a_{35} + a_{11}a_{25}a_{33} - a_{12}a_{25}a_{34} - a_{15}a_{21}a_{33} + a_{15}a_{22}a_{34} + a_{11}a_{23}a_{36} \\
+ a_{12}a_{23}a_{36} - a_{22}a_{13}a_{36} - a_{21}a_{14}a_{36} \\
g_5 &= -a_{24}a_{16}a_{32} - a_{25}a_{13}a_{32} + a_{14}a_{26}a_{32} + a_{15}a_{23}a_{32} - a_{12}a_{26}a_{34} + a_{22}a_{16}a_{34} - a_{12}a_{23}a_{35} \\
+ a_{22}a_{13}a_{35} + a_{12}a_{25}a_{33} - a_{15}a_{22}a_{33} + a_{12}a_{24}a_{36} - a_{14}a_{22}a_{36} \\
g_6 &= -a_{25}a_{16}a_{31} + a_{15}a_{26}a_{31} - a_{11}a_{26}a_{35} + a_{21}a_{16}a_{35} + a_{11}a_{25}a_{36} - a_{15}a_{21}a_{36} \\
g_7 &= -a_{25}a_{16}a_{32} + a_{15}a_{26}a_{32} - a_{12}a_{26}a_{35} + a_{22}a_{16}a_{35} + a_{12}a_{25}a_{36} - a_{15}a_{22}a_{36}
\end{align*}
\]
In the final step, we employ Sylvester resultant to eliminate one of the remaining two variables from equation (9) and (5). The Sylvester resultant is the determinant of the coefficient matrix of the following system.

\[
\begin{bmatrix}
g_3 & g_4 s + g_1 + g_6 & g_5 s^2 + (g_2 + g_7) s \\
0 & g_3 & g_4 s + g_1 + g_6 \\
1 & 0 & g_5 s^2 + (g_2 + g_7) s \\
0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c^3 \\
c^2 \\
c \\
1 \\
\end{bmatrix} = 0 \quad (10)
\]

The univariate polynomial is

\[
\sum_{i=0}^{4} h_i s^i = 0 \quad (11)
\]

where

\[
\begin{align*}
h_0 &= -g_2^2 - 2g_1 g_6 - g_6^2 + g_7^2 \\
h_1 &= 2g_3g_2 - 2g_4g_6 + 2g_3g_7 - 2g_4g_1 \\
h_2 &= -g_4^2 + g_1^2 + g_6^2 + g_2^2 + g_7^2 - 2g_3^2 + 2g_1g_6 + 2g_2g_7 + 2g_3g_5 \\
h_3 &= 2g_4g_1 + 2g_4g_6 + 2g_5g_2 + 2g_5g_7 - 2g_3g_2 - 2g_3g_7 \\
h_4 &= g_4^2 + g_6^2 + g_5^2 - 2g_3g_5
\end{align*}
\]

Hence we have 4 solutions for \( s \) generically. Back-substituting the solutions of \( s \) into equation (9), we compute the corresponding solutions for \( c \). Each solution of \( s \) corresponds to one solution for \( c \), because we can reduce the order of equation (9) to linear in \( c \) after \( s \) is known by replacing the quadratic terms of \( c^2 \) with \( 1 - s^2 \).

\[
c = \frac{(g_3 - g_5)s^2 - (g_2 + g_7)s - g_3}{g_4 s + g_1 + g_6} \quad (12)
\]

After \( c \) and \( s \) is determined, we can compute the corresponding solutions for \( x \) and \( y \) using (7), and we get a total of 4 solutions.