Numerical inversion of the Laplace transform via regularized analytic continuation

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Outline

- Laplace transformation and analytic continuation
- Smoothing transformation
- Analtytic continuation by rotation
- Inverse Laplace transformation via analytic continuation
- Analysis and comparison of real methods of inverse Laplace transformation
- Limitations of the most effective real method
- Q & A. References

Laplace transformation and Analytic continuation

Laplace transform

$$F(p) = \int_0^\infty f(t)\mathrm{e}^{-pt}\mathrm{d}t \tag{1}$$

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{pt}F(p)\,\mathrm{d}p \tag{2}$$

- Real inversion formula: $f(t) = \mathcal{L}^{-1}[F(p)]$ is unknown.

Smoothing transformation

$$F_R(c+p) = \int_0^\infty F(c+px)\delta_R(x-1)\,\mathrm{d}x \tag{3}$$

- p and c are complex parameters
- $\delta_R(x-1) \longrightarrow \delta(x-1)$ as $R \to \infty$
- $F_R(c+p) \longrightarrow F(c+p)$ as $R \to \infty$
- if $w(x)$ is an arbitrary function, such that $w(1) = 1$, then $\tilde{\delta}_R(x-1) = w(x)\delta_R(x-1) \longrightarrow \delta(x-1)$ as $R \to \infty$

Analytic continuation by rotation

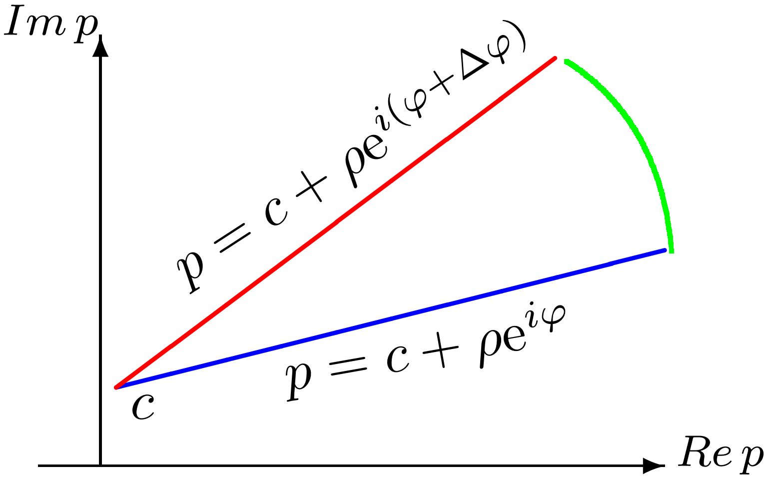

Figure 1

$$F_R(c+\rho e^{i(\varphi+\Delta\varphi)}) = \int_0^\infty F(c+\rho e^{i(\varphi+\Delta\varphi)}z)\delta_R(z-1)\,\mathrm{d}z =$$

$$= \int_0^{\infty\, e^{-i\Delta\varphi}} F(c+\rho e^{i(\varphi+\Delta\varphi)}z)\delta_R(z-1)\,\mathrm{d}z =$$

$$= e^{-i\Delta\varphi}\int_0^\infty F(c+\rho e^{i\varphi}u)\delta_R(ue^{-i\Delta\varphi}-1)\,\mathrm{d}u$$

Theorems

Theorem 1 *If for a given complex parameter* p, $|p| \neq 0$, *the function* $F(c+p)$ *is analytically continued from a ray* $p = \rho \mathrm{e}^{i\varphi}$ *onto the ray* $p = \rho \mathrm{e}^{i(\varphi + \Delta\varphi)}$, *then*

$$F_R(c + p\mathrm{e}^{i\Delta\varphi}) = \mathrm{e}^{-i\Delta\varphi} \int_0^\infty F(c+pu)\delta_R(u\mathrm{e}^{-i\Delta\varphi})\,\mathrm{d}u \qquad (4)$$

and

$$\lim_{R\to\infty} F_R(c + p\mathrm{e}^{i\Delta\varphi}) = F(c + p\mathrm{e}^{i\Delta\varphi})$$

Theorem 2 *An operator* $\mathcal{A}_R : F(c+p) \longrightarrow F_R(c + p\mathrm{e}^{i\Delta\varphi})$ *defined by equation (4) is a regularizing operator of analytic continuation.*

Parameter of regularization

$$\delta_R(x-1) = \frac{\sin\left(R \ln x\right)}{\pi(x-1)} \tag{5}$$

$$F_R(c + p\mathrm{e}^{i\Delta\varphi}) = \frac{1}{\pi}\int_0^\infty F(c+pu)\frac{\sin R\ln(u\mathrm{e}^{-i\Delta\varphi})}{u - \mathrm{e}^{i\Delta\varphi}}\,\mathrm{d}u \tag{6}$$

- integrand increases with R as $\left|\sin\left(R\ln(u\mathrm{e}^{-i\Delta\varphi})\right)\right| \sim \sinh(R\Delta\varphi)$
- R can be increased as lond as computational error $\epsilon\sinh(R\Delta\varphi)$ remains relatively small
- for double precision of input data and $\Delta\varphi = \pi/2 \;\; R_{max} \approx 20$

Numerical example

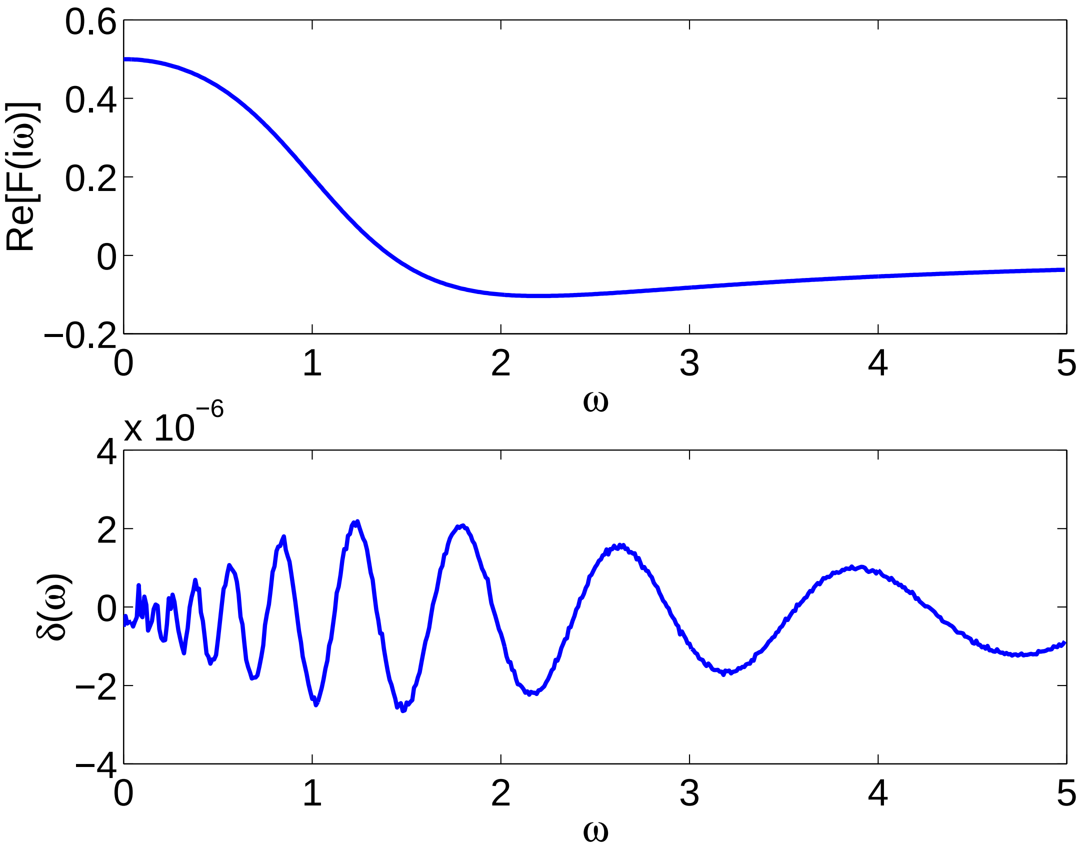

Figure 2. Analytic continuation of $F(p) = 1/((1+p)^2+1)$ from the real onto the imaginary axis (upper frame) and its absolute errors (lower frame). $R_{opt} = 15.6, \quad \delta_R(x-1) = 2x\sin(R\ln x)/\pi(x^2-1)$

Inversion of the Laplace transforms via regularized analytic continuation

$$f(t) = \mathcal{L}^{-1}[F(p)], \qquad f_R(t) = \mathcal{L}^{-1}[F_R(p)]$$

$$f_R(t) = \int_0^\infty F(x)\Pi(R, tx)\mathrm{d}x \tag{7}$$

$$\Pi(R, x) = \mathcal{L}^{-1}[\delta_R(p-1)] \tag{8}$$

$$\delta_R(p-1) = \int_0^\infty \mathrm{e}^{-px}\Pi(R, x)\,\mathrm{d}x \tag{9}$$

$$f_R(t) = \int_0^\infty f(tx)\delta_R(x-1)\mathrm{d}x \tag{10}$$

$$F_R(p) = \int_0^\infty F(px)x^{-1}\delta_R(x^{-1}-1)\mathrm{d}x \tag{11}$$

Common features

- Basic properties of the Laplace transform are fullfilled, at least approximately
- There is no exact real inversion formula of the convolution type
- The approximate inverse Laplace transform of $F(p) = \exp(-px)$ is $\delta_R(x/t - 1)/t$
- The function $\delta_R(x/t - 1)/t$ determines all features of a particular method
- In general, the accuracy of $f_R(t)$ decreases with time

Gaver's method

$$\delta_N(x-1) = \frac{(2N)!\ln 2}{N!(N-1)!}\left(2^{-x} - 2^{-2x}\right)^N \tag{12}$$

$$f_N(t) = \frac{(2N)!}{N!(N-1)!}\frac{\ln 2}{t}\sum_{k=0}^{N}(-1)^k\binom{N}{k}F\left(\frac{\ln 2(N+k)}{t}\right) \tag{13}$$

$$N_{max} = 22$$

Gaver's and Stehfest's approximations for delta-function, $N = 20$

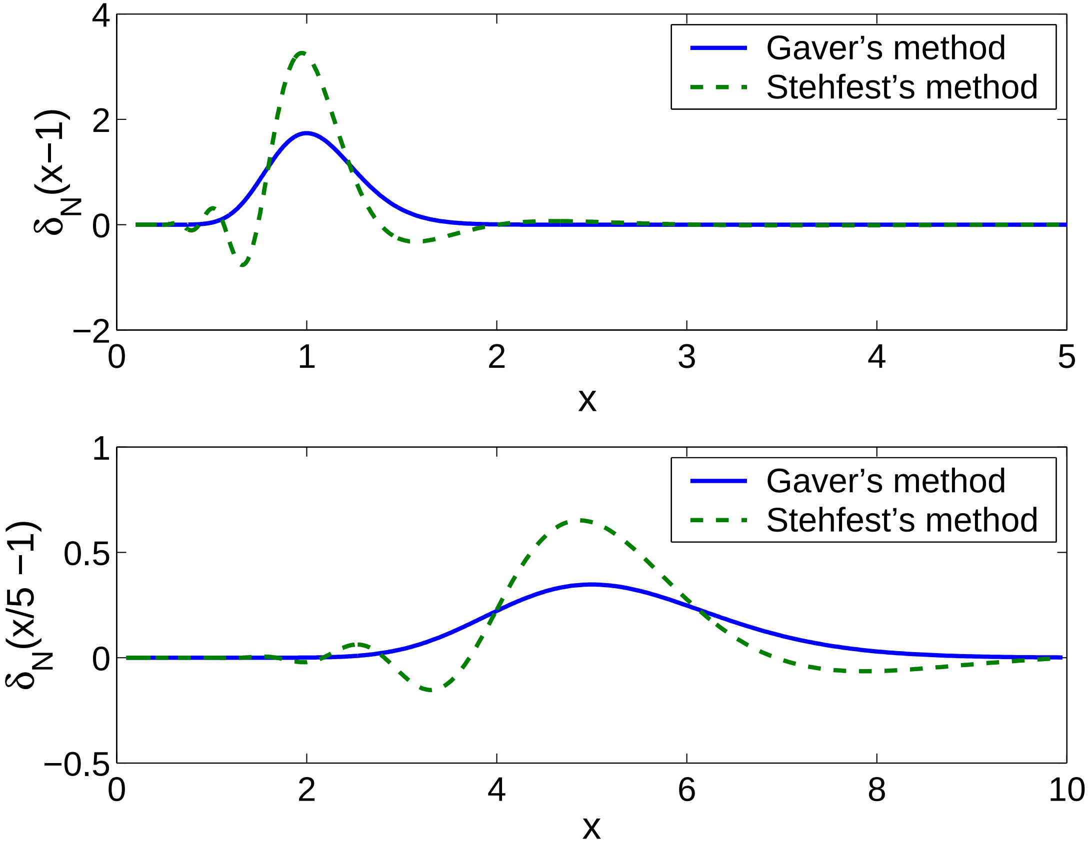

Figure 3. The delta function approximation for Gaver's method and $\mathcal{L}^{-1}[\exp(-pt); x]$ for Stehfest's method. $t = 1$ (upper frame) and $t = 5$ (lower frame) and $N = 20$.

Post-Widder's formula

$$\delta_N(x-1) = N^{N+1} x^N \exp(-Nx)/N! \tag{14}$$

$$\Pi(N,x) = \frac{N^{N+1}}{N!}\mathcal{L}^{-1}[p^N \exp(-Np)] = \frac{N^{N+1}}{N!}\delta^{(N)}(x-N) \tag{15}$$

$$f_N(t) = \frac{(-1)^N}{N!}\frac{N^{N+1}}{t^{N+1}} F^{(N)}\left(N/t\right) \tag{16}$$

$$N_{max} = 30$$

Analysis of a potential method

$$\delta_R(x) = \frac{R\exp(-R^2x^2)}{\sqrt{\pi}} \tag{17}$$

$$\delta_R(x-1) = \frac{R\exp(-R^2\ln^2 x)}{\sqrt{\pi}} \tag{18}$$

$$F_R(i\omega) = \frac{-iR}{\sqrt{\pi}}\int_0^\infty F(u\omega)\exp\left(-R^2\ln^2(-iu)\right)\mathrm{d}u. \tag{19}$$

$$\exp\left(-R^2\ln^2(-iu)\right) \sim \exp(R^2\pi^2/4)$$

$$R_{max} \approx 3.5$$

Delta-function approximations

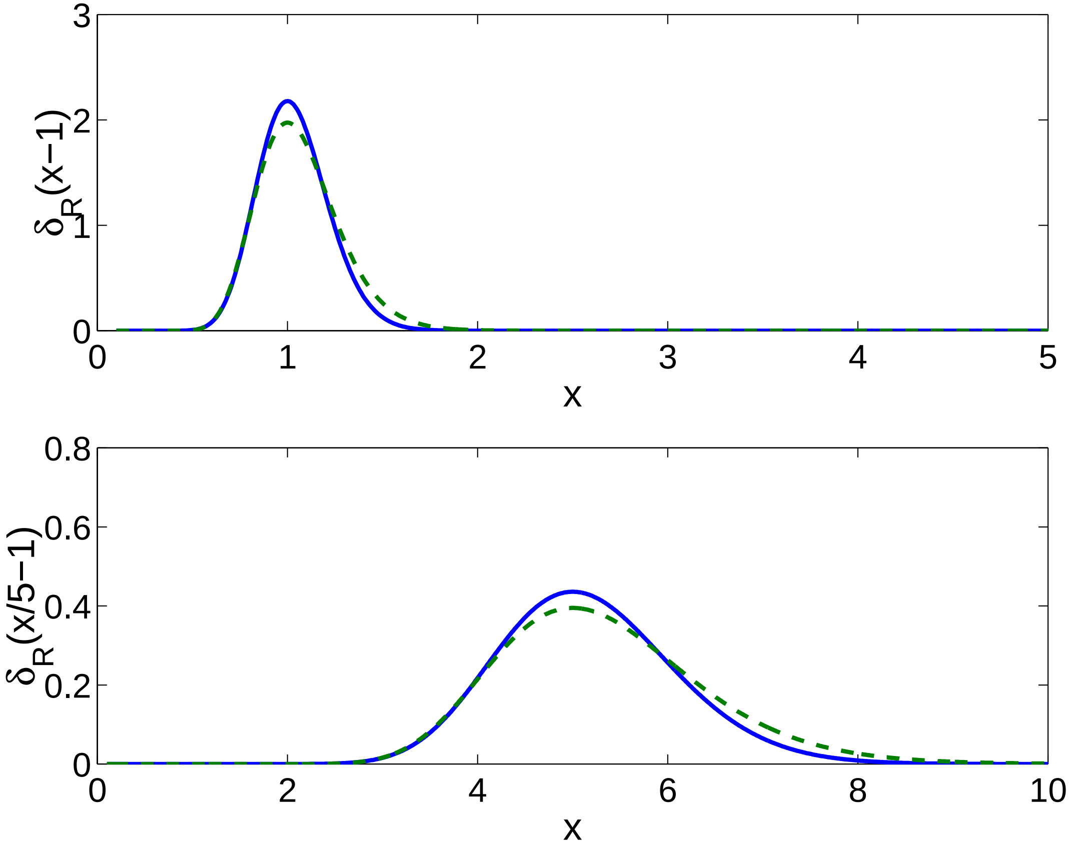

Figure 4. The delta function approximations for Post-Widder method ($N = 30$, solid line) and $y = R\exp\left(-R^2\ln^2(x/t)\right)/t\sqrt{\pi}$ ($R = 3.5$, dash line). $t = 1$ (upper frame) and $t = 5$ (lower frame).

Legendre expansion method

$$f_N(t) = \sum_{n=0}^{N} c_n P_n(1 - 2\mathrm{e}^{-\sigma t}) \tag{20}$$

$$c_n = \sigma(2n+1) \sum_{m=0}^{n} \frac{(-1)^m (n+m)!}{(n-m)!(m!)^2} F(\sigma(m+1)).$$

$$\sigma = \ln 2/2t$$

$$\sigma = const$$

Delta-function approximation for Legendre expansion method $\sigma = \ln 2/2t$

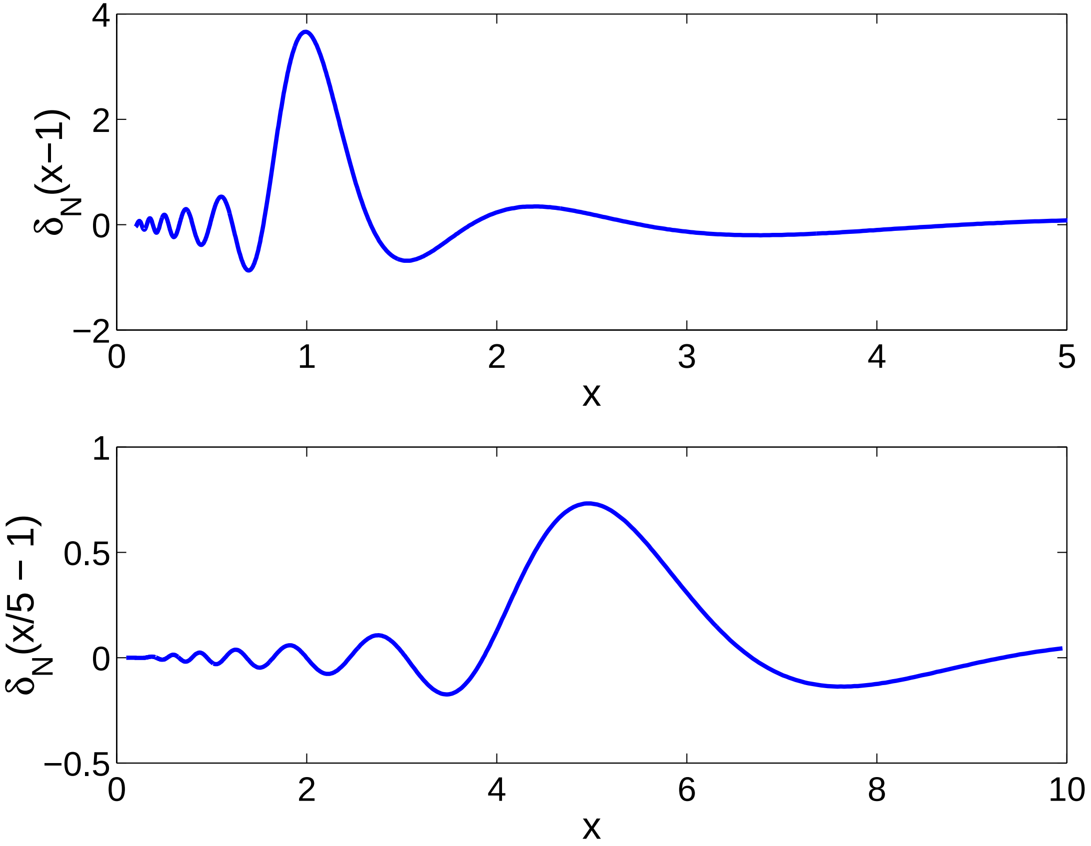

Figure 5. The inverse Laplace transform of $\exp(-xp)$ via the Legendre expansion method when $x = 1$ (upper frame) and $x = 5$ (lower frame); $N = 20$

Delta-function approximation for Legendre expansion method $\sigma = 1$

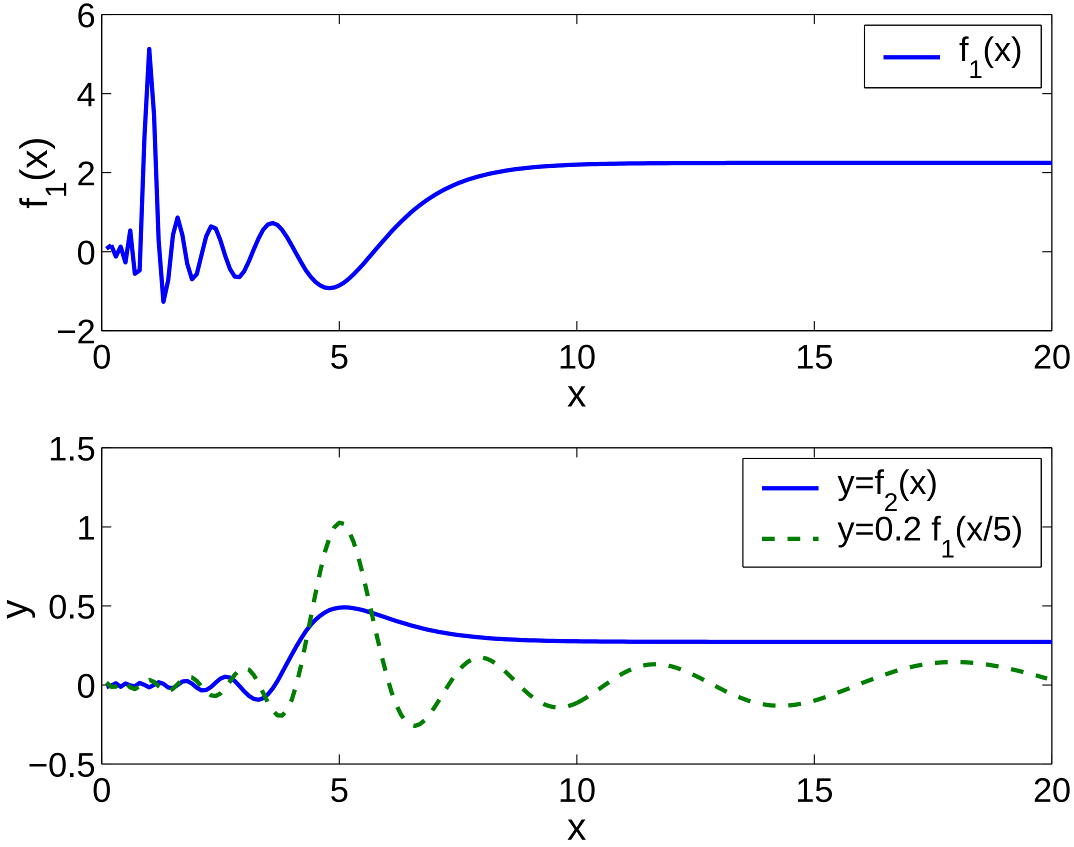

Figure 6. The inverse Laplace transforms $f_1(t) = \mathcal{L}^{-1}[\exp(-p)]$ and $f_2(t) = \mathcal{L}^{-1}[\exp(-5p)]$

obtained using the Legendre expansion method for $N = 20$

The most effective method

$$\delta_R(x-1) = \frac{\mathrm{e}^{\alpha(1-x)}}{\pi} \frac{x^a \sin(R\ln x)}{x-1} \qquad (21)$$

$$R_{max} \approx 20$$

$$\Pi(R,x) = \frac{\mathrm{e}^{\alpha}}{\pi^2} \mathrm{Im}\left[\sin\pi(a+iR)\Gamma(a+iR) \right. \\ \left. \times\, x^{-a-iR} {}_1F_1\,(1;1-a-iR;x)\right] \qquad (22)$$

where ${}_1F_1(a;b;z)$ is confluent hypergeometric function, and $a<1$, $\alpha \geq 0$

The most ‘focusing’ approximation to delta-function

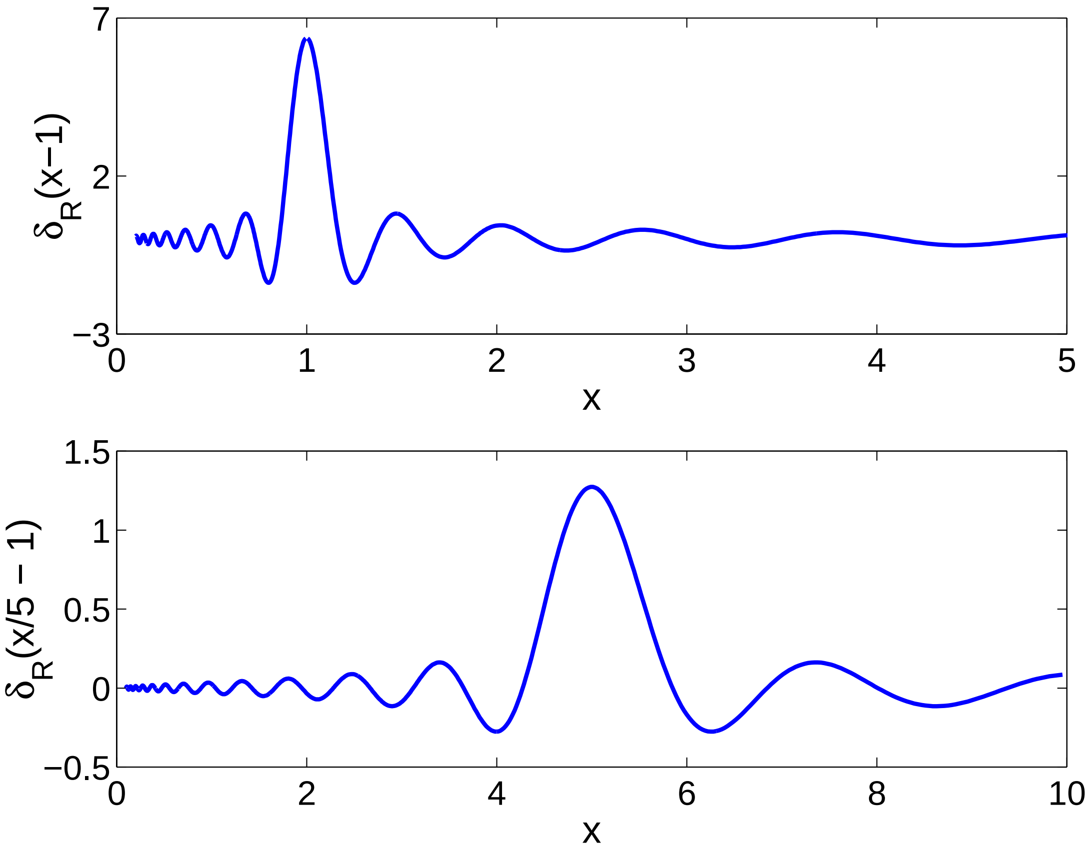

Figure 7. The function $y = \delta_R(x/t-1)/t = \pi^{-1}\sqrt{x/t}(x-t)^{-1}\sin\left(R\ln(x/t)\right)$ for $t=1$ (upper frame) and $t=5$ (lower frame) where $R=20$.

Limitations of the most effective method

$$f_R(t) = \frac{e^\alpha}{\pi} \int_0^\infty f(tu) u^a e^{-\alpha u} \frac{\sin R \ln u}{u-1} \, du \qquad (23)$$

- Functions of type $\varphi_1(t) = t^r$, $r > -1$ and $\varphi_2(t) = t^s \ln t$, $s > -1$ can be found accurately through inverse Laplace transformation for any $t > 0$
- Difference between $f(t) = t^r \exp(-zt)$ (where z is complex parameter) and $f_R(t)$ increases as the oscillatory portion grows.
- Two consecutive extrema of the exact inverse transform located at t_1 and $t_2 > t_1$, can be resolved only if $t_2/t_1 > \exp(\pi/R)$
- The resolution of frequency ω_0 may be guaranteed with some degree of confidence only for $t < t_{max} = R/\omega_0$

Restoration of $\sin t$ from its real-valued Laplace transform

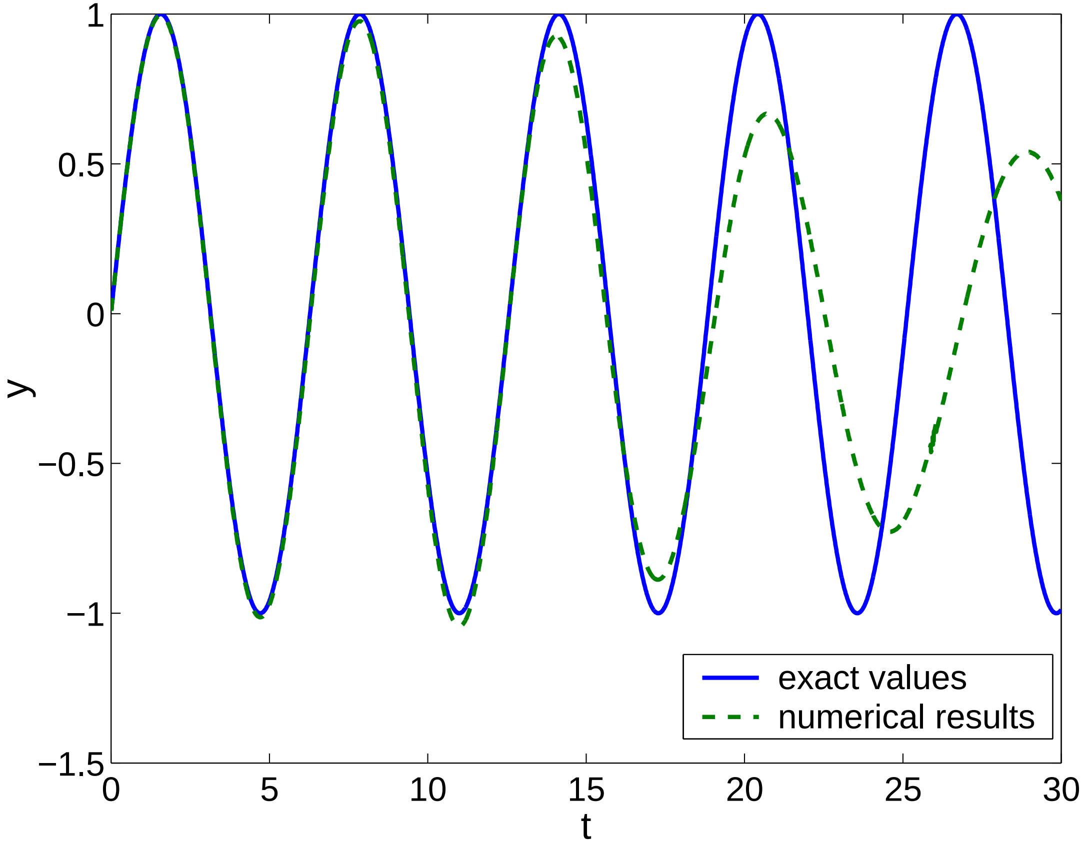

Figure 8. Numerical inversion of $F(p) = 1/(1+p^2)$ for $R = 20$, $a = -1.8$, and $\alpha = 1.8$

Conclusion

- There exists no exact formula of the convolution type for inverting real-valued Laplace transforms.
- Analytic continuation by rotation provides a unifying procedure for deriving several known and possibly new real methods
- The differences between the various methods are determined by underlying approximation for $\delta(x-1)$. The potential of various real methods can be compared by analyzing the ‘ focusing’ abilities of the approximations $\delta_R(x-1)$.
- Real-valued method that is based on the approximation $\delta_R(x-1) = \mathrm{e}^{\alpha(1-x)} x^a \sin(R \ln x)/(x-1)\pi$ is has the least limitations and leads to the most accurate results.

Q & A

References

- Kryzhniy V. V. On regularization method for numerical inversion of Laplace transforms, *J. of Inverse and Ill-posed probl.*, 2004, v. 12, p. 279–96
- Kryzhniy V. V. Numerical inversion of the Laplace transform: analysis via regularized analytic continuation, *Inverse probl.*, 2006, v. 22, p. 579–97
- http://www.laplacetransform.org