MATH 4512. Differential Equations with Applications.
Midterm Exam #2. April 6, 2016.
Problems and Solutions.

1. Verify whether of not the functions

$$f_1(t) = t, \quad f_2(t) = e^t, \quad f_3(t) = \ln t, \quad \text{and} \quad f_4(t) = \tan t$$

are linearly independent on $(0, \pi/2)$.

Solution. Suppose that

$$c_1 t + c_2 e^t + c_3 \ln t + c_4 \tan t \equiv 0 \quad \text{on} \quad (0, \pi/2) \quad \text{with some constants} \quad c_1, c_2, c_3, c_4.$$

By considering limits as $t \to \pi/2$, and then $t \to 0$, we see that $c_4 = c_3 = 0$. Now we have $c_1 t + c_2 e^t \equiv 0$. Differentiating this equality twice, we get $c_2 = 0$, and finally, $c_1 = 0$. This means that the given functions are **linearly independent**.

2. Find a particular solution to the equation

$$Ly = t^2 y'' - t y' + y = 4t \cdot \ln t \quad \text{for} \quad t > 0,$$

given that $y_1(t) = t$ and $y_2(t) = t \cdot \ln t$ are linearly independent solutions to the homogeneous equation $Ly = 0$.

Solution. The general solution has the form $y = c_1 y_1 + c_2 y_2 + Y$. Here a particular solution Y can be found by the method of of variation of parameters in the form $Y = u_1 y_1 + u_2 y_2$, where

$$u_1' y_1 + u_2' y_2 = 0, \qquad u_1' y_1' + u_2' y_2' = 4t^{-1} \ln t.$$

Here the right hand side is $4t \ln t$ divided by t^2 – the coefficient of y''. After a minor simplification, we rewrite this system in the form

$$u_1' + u_2' \ln t = 0, \qquad u_1' + u_2'(1 + \ln t) = 4t^{-1} \ln t.$$

Subtracting both sides of the first equation from the second one, we get:

$$u_2' = 4t^{-1} \ln t, \qquad u_1' = -4t^{-1} (\ln t)^2.$$

Both u_1 and u_2 are obtained by integration with substitution $u = \ln t,\ du = t^{-1} dt$. We do not need to add the constants of integration, because eventually they will be added to the constants c_1 and c_2 in $y = c_1 y_1 + c_2 y_2 + Y$. Then

$$\begin{aligned} u_1 &= -4 \int \frac{(\ln t)^2}{t} dt = -\frac{4}{3} (\ln t)^3, \quad u_2 = 4 \int \frac{\ln t}{t} dt = 2 (\ln t)^2, \\ Y &= u_1 y_1 + u_2 y_2 = -\frac{4t}{3} (\ln t)^3 + 2t (\ln t)^3 = \frac{2t}{3} (\ln t)^3. \end{aligned}$$

Finally, the general solution is

$$y = c_1 y_1 + c_2 y_2 + Y = (c_1 + c_2 \ln t) t + \frac{2t}{3} (\ln t)^3.$$

3. Find the general solution of the equation

$$y''' - 9y'' + 25y' - 17y = 0.$$

Solution. The characteristic equation $\chi(r) = r^3 - 9r^2 + 25r - 17 = 0$ has a root $r_1 = 1$. We can rewrite it in the form

$$\chi(r) = (r-1)(r^2 - 8r + 17) = (r-1)\big[(r-4)^2 + 1\big] = 0,$$

so that the remaining two root $r_{2,3} = 4 \pm i$. Correspondingly, the general solution is

$$y = c_1 e^t + e^{4t}(c_2 \cos t + c_3 \sin t).$$

4. Find the general solution of the equation

$$y'' + y = 4(t-1)\cos t.$$

Solution. The characteristic equation $r^2 + 1 = 0$ has roots $r_{1,2} = \pm i$, and the general solution

$$y = c_1 \cos t + c_2 \sin t + Y, \quad \text{where} \quad (D^2+1)Y = Y'' + Y = 4(t-1)\cos t.$$

Having in mind that $e^{it} = \cos t + i \sin t$, we can find Y as the real part of a complex solution of $(D^2+1)Z = 4(t-1)e^{it}$. Since i is one of roots $r_{1,2}$, we have $Z = e^{it} t(At + B)$ with some constants A and B. Then

$$(D^2+1)Z = e^{it}\big[(D+i)^2 + 1\big](At^2 + Bt) = e^{it}(D^2 + 2iD)(At^2 + Bt) = e^{it}(4t-4).$$

Comparing the coefficients in both sides of $(D^2 + 2iD)(At^2 + Bt) = 4t - 4$, we get

$$4iA = 4, \quad 2A + 2iB = -4, \quad \text{so that} \quad A = -i, \quad B = 1 + 2i,$$

and

$$Y = \mathrm{Re}\big[t(-it + 1 + 2i)(\cos t + i \sin t)\big] = t \cdot \big[\cos t + (t-2)\sin t\big].$$