# MATH 4512. Differential Equations with Applications.
# Midterm Exam #1. February 24, 2016.
# Problems and Solutions

**1.** Solve the initial value problem

$$x\frac{dy}{dx} = 2y + \frac{3}{x}, \qquad y(1) = 0.$$

**Solution.** After dividing by $x \neq 0$, we get a linear equation

$$Ly = y' + py = g \quad \text{with} \quad p(x) = -2 \cdot x^{-1}, \quad g(x) = 3 \cdot x^{-2}.$$

The integrating factor (see p. 36)

$$\mu(x) = \exp\Big(\int p(x)\,dx\Big) = \exp(-2\ln x) = x^{-2} \quad \text{satisfies} \quad \mu' = p\mu,$$

so that the equation is reduced to the form

$$\mu(y' + py) = (\mu y)' = \mu g = 3 \cdot x^{-4} \implies \mu y = \int 3 \cdot x^{-4}\,dx = -x^{-3} + C.$$

Therefore, the general solution to the given equation is

$$y = \mu^{-1} \cdot (-x^{-3} + C) = -x^{-1} + Cx^2.$$

Note that the constant $C$ may be different on the sets $\{x > 0\}$ and $\{x < 0\}$. The initial condition $y(1) = 0$ belongs to the set $\{x > 0\}$. In this set, we must have $C = 1$, and finally,

$$y(x) = x^2 - x^{-1} \quad \text{for} \quad x > 0.$$

**2.** Find the general solution of the equation

$$\frac{dy}{dx} - xy^2 = xy.$$

**Solution.** We have for $y \neq 0,\ y \neq -1$:

$$\frac{dy}{dx} = x(y^2 + y) \implies \frac{dy}{y(y+1)} = x\,dx \implies \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = x\,dx$$

$$\implies \ln\left|\frac{y}{y+1}\right| = \ln|y| - \ln|y+1| = \frac{x^2}{2} + C_0 \implies \frac{y}{y+1} = \pm e^{C_0} \cdot e^{x^2/2}.$$

This argument does not work for $y = 0$ and $y = -1$, because the denominator $y(y+1) = 0$. We need to verify these cases separately. Both $y = 0$ and $y = -1$ are solutions, hence we finally get

$$\frac{y}{y+1} = C \cdot e^{x^2/2}, \quad \text{and} \quad y \equiv -1.$$

**3.** Find the general solution of the equation

$$x\frac{dy}{dx} = y(\ln y - \ln x).$$

**Solution.** This is a homogeneous equation.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x}, \quad \text{substitute} \quad y = xv \quad \Longrightarrow v + x\frac{dv}{dx} = v \ln v, \quad \frac{dv}{v(\ln v - 1)} = \frac{dx}{x}, \\ \ln|\ln v - 1| &= \ln|x| + C_0, \quad \ln v - 1 = Cx, \quad y = xv = x\,e^{Cx+1} \quad \text{for} \quad x > 0. \end{aligned}$$

**4.** Solve the initial value problem

$$y'' - 7y' + 6y = 0, \qquad y(0) = 2,\ y'(0) = 7.$$

**Solution.** The characteristic equation

$$r^2 - 7r + 6 = 0 \quad \text{has two distinct roots} \quad r_1 = 1, \quad r_2 = 6.$$

Therefore, the general solution is

$$y(x) = c_1 e^x + c_2 e^{6x}.$$

The initial conditions give us

$$y(0) = c_1 + c_2 = 2, \qquad y'(0) = c_1 + 6c_2 = 7.$$

Hence $c_1 = c_2 = 1$, and finally,

$$y(x) = e^x + e^{6x}.$$