

# Contents

<b>Preface to the Second Edition</b>	<b>xiii</b>
<b>Preface to the First Edition</b>	<b>xvii</b>
<b>1 Background in Linear Algebra</b>	<b>1</b>
1.1 Matrices . . . . .	1
1.2 Square Matrices and Eigenvalues . . . . .	2
1.3 Types of Matrices . . . . .	4
1.4 Vector Inner Products and Norms . . . . .	5
1.5 Matrix Norms . . . . .	7
1.6 Subspaces, Range, and Kernel . . . . .	9
1.7 Orthogonal Vectors and Subspaces . . . . .	10
1.8 Canonical Forms of Matrices . . . . .	14
1.8.1 Reduction to the Diagonal Form . . . . .	15
1.8.2 The Jordan Canonical Form . . . . .	15
1.8.3 The Schur Canonical Form . . . . .	16
1.8.4 Application to Powers of Matrices . . . . .	18
1.9 Normal and Hermitian Matrices . . . . .	20
1.9.1 Normal Matrices . . . . .	20
1.9.2 Hermitian Matrices . . . . .	23
1.10 Nonnegative Matrices, $M$ -Matrices . . . . .	25
1.11 Positive Definite Matrices . . . . .	29
1.12 Projection Operators . . . . .	32
1.12.1 Range and Null Space of a Projector . . . . .	32
1.12.2 Matrix Representations . . . . .	34
1.12.3 Orthogonal and Oblique Projectors . . . . .	34
1.12.4 Properties of Orthogonal Projectors . . . . .	36
1.13 Basic Concepts in Linear Systems . . . . .	37
1.13.1 Existence of a Solution . . . . .	37
1.13.2 Perturbation Analysis . . . . .	38
Exercises . . . . .	39
Notes and References . . . . .	43

<b>2</b>	<b>Discretization of Partial Differential Equations</b>	<b>45</b>
2.1	Partial Differential Equations . . . . .	45
2.1.1	Elliptic Operators . . . . .	46
2.1.2	The Convection Diffusion Equation . . . . .	48
2.2	Finite Difference Methods . . . . .	48
2.2.1	Basic Approximations . . . . .	48
2.2.2	Difference Schemes for the Laplacian Operator . . . . .	50
2.2.3	Finite Differences for One-Dimensional Problems . . . . .	51
2.2.4	Upwind Schemes . . . . .	52
2.2.5	Finite Differences for Two-Dimensional Problems . . . . .	54
2.2.6	Fast Poisson Solvers . . . . .	55
2.3	The Finite Element Method . . . . .	60
2.4	Mesh Generation and Refinement . . . . .	66
2.5	Finite Volume Method . . . . .	68
	Exercises . . . . .	71
	Notes and References . . . . .	72
<b>3</b>	<b>Sparse Matrices</b>	<b>73</b>
3.1	Introduction . . . . .	73
3.2	Graph Representations . . . . .	75
3.2.1	Graphs and Adjacency Graphs . . . . .	75
3.2.2	Graphs of PDE Matrices . . . . .	76
3.3	Permutations and Reorderings . . . . .	77
3.3.1	Basic Concepts . . . . .	77
3.3.2	Relations with the Adjacency Graph . . . . .	79
3.3.3	Common Reorderings . . . . .	81
3.3.4	Irreducibility . . . . .	89
3.4	Storage Schemes . . . . .	89
3.5	Basic Sparse Matrix Operations . . . . .	92
3.6	Sparse Direct Solution Methods . . . . .	93
3.6.1	MD Ordering . . . . .	93
3.6.2	ND Ordering . . . . .	94
3.7	Test Problems . . . . .	94
	Exercises . . . . .	97
	Notes and References . . . . .	101
<b>4</b>	<b>Basic Iterative Methods</b>	<b>103</b>
4.1	Jacobi, Gauss–Seidel, and Successive Overrelaxation . . . . .	103
4.1.1	Block Relaxation Schemes . . . . .	106
4.1.2	Iteration Matrices and Preconditioning . . . . .	110
4.2	Convergence . . . . .	111
4.2.1	General Convergence Result . . . . .	112
4.2.2	Regular Splittings . . . . .	115
4.2.3	Diagonally Dominant Matrices . . . . .	116
4.2.4	Symmetric Positive Definite Matrices . . . . .	119
4.2.5	Property A and Consistent Orderings . . . . .	119

4.3	Alternating Direction Methods . . . . .	124
	Exercises . . . . .	126
	Notes and References . . . . .	128
<b>5</b>	<b>Projection Methods</b>	<b>129</b>
5.1	Basic Definitions and Algorithms . . . . .	129
5.1.1	General Projection Methods . . . . .	130
5.1.2	Matrix Representation . . . . .	131
5.2	General Theory . . . . .	132
5.2.1	Two Optimality Results . . . . .	133
5.2.2	Interpretation in Terms of Projectors . . . . .	134
5.2.3	General Error Bound . . . . .	135
5.3	One-Dimensional Projection Processes . . . . .	137
5.3.1	Steepest Descent . . . . .	138
5.3.2	MR Iteration . . . . .	140
5.3.3	Residual Norm Steepest Descent . . . . .	142
5.4	Additive and Multiplicative Processes . . . . .	143
	Exercises . . . . .	145
	Notes and References . . . . .	149
<b>6</b>	<b>Krylov Subspace Methods, Part I</b>	<b>151</b>
6.1	Introduction . . . . .	151
6.2	Krylov Subspaces . . . . .	152
6.3	Arnoldi's Method . . . . .	153
6.3.1	The Basic Algorithm . . . . .	154
6.3.2	Practical Implementations . . . . .	156
6.4	Arnoldi's Method for Linear Systems . . . . .	159
6.4.1	Variation 1: Restarted FOM . . . . .	160
6.4.2	Variation 2: IOM and DIOM . . . . .	161
6.5	Generalized Minimal Residual Method . . . . .	164
6.5.1	The Basic GMRES Algorithm . . . . .	164
6.5.2	The Householder Version . . . . .	165
6.5.3	Practical Implementation Issues . . . . .	167
6.5.4	Breakdown of GMRES . . . . .	171
6.5.5	Variation 1: Restarting . . . . .	171
6.5.6	Variation 2: Truncated GMRES Versions . . . . .	172
6.5.7	Relations Between FOM and GMRES . . . . .	177
6.5.8	Residual Smoothing . . . . .	181
6.5.9	GMRES for Complex Systems . . . . .	184
6.6	The Symmetric Lanczos Algorithm . . . . .	185
6.6.1	The Algorithm . . . . .	185
6.6.2	Relation to Orthogonal Polynomials . . . . .	186
6.7	The Conjugate Gradient Algorithm . . . . .	187
6.7.1	Derivation and Theory . . . . .	187
6.7.2	Alternative Formulations . . . . .	191
6.7.3	Eigenvalue Estimates from the CG Coefficients . . . . .	192

6.8	The Conjugate Residual Method . . . . .	194
6.9	Generalized Conjugate Residual, ORTHOMIN, and ORTHODIR . . . . .	194
6.10	The Faber–Manteuffel Theorem . . . . .	196
6.11	Convergence Analysis . . . . .	198
6.11.1	Real Chebyshev Polynomials . . . . .	199
6.11.2	Complex Chebyshev Polynomials . . . . .	200
6.11.3	Convergence of the CG Algorithm . . . . .	203
6.11.4	Convergence of GMRES . . . . .	205
6.12	Block Krylov Methods . . . . .	208
	Exercises . . . . .	212
	Notes and References . . . . .	215
<b>7</b>	<b>Krylov Subspace Methods, Part II</b> . . . . .	<b>217</b>
7.1	Lanczos Biorthogonalization . . . . .	217
7.1.1	The Algorithm . . . . .	217
7.1.2	Practical Implementations . . . . .	220
7.2	The Lanczos Algorithm for Linear Systems . . . . .	221
7.3	The Biconjugate Gradient and Quasi-Minimal Residual Algorithms . . . . .	222
7.3.1	The BCG Algorithm . . . . .	222
7.3.2	QMR Algorithm . . . . .	224
7.4	Transpose-Free Variants . . . . .	228
7.4.1	CGS . . . . .	229
7.4.2	BICGSTAB . . . . .	231
7.4.3	TFQMR . . . . .	234
	Exercises . . . . .	241
	Notes and References . . . . .	243
<b>8</b>	<b>Methods Related to the Normal Equations</b> . . . . .	<b>245</b>
8.1	The Normal Equations . . . . .	245
8.2	Row Projection Methods . . . . .	247
8.2.1	Gauss–Seidel on the Normal Equations . . . . .	247
8.2.2	Cimmino’s Method . . . . .	249
8.3	Conjugate Gradient and Normal Equations . . . . .	251
8.3.1	CGNR . . . . .	252
8.3.2	CGNE . . . . .	253
8.4	Saddle-Point Problems . . . . .	254
	Exercises . . . . .	257
	Notes and References . . . . .	259
<b>9</b>	<b>Preconditioned Iterations</b> . . . . .	<b>261</b>
9.1	Introduction . . . . .	261
9.2	Preconditioned Conjugate Gradient . . . . .	262
9.2.1	Preserving Symmetry . . . . .	262
9.2.2	Efficient Implementations . . . . .	265

9.3	Preconditioned Generalized Minimal Residual . . . . .	267
9.3.1	Left-Preconditioned GMRES . . . . .	268
9.3.2	Right-Preconditioned GMRES . . . . .	269
9.3.3	Split Preconditioning . . . . .	270
9.3.4	Comparison of Right and Left Preconditioning . . . . .	271
9.4	Flexible Variants . . . . .	272
9.4.1	FGMRES . . . . .	273
9.4.2	DQGMRES . . . . .	276
9.5	Preconditioned Conjugate Gradient for the Normal Equations . . . . .	276
9.6	The Concus, Golub, and Widlund Algorithm . . . . .	278
	Exercises . . . . .	279
	Notes and References . . . . .	281
<b>10</b>	<b>Preconditioning Techniques</b> . . . . .	<b>283</b>
10.1	Introduction . . . . .	283
10.2	Jacobi, Successive Overrelaxation, and Symmetric Successive Overrelaxation Preconditioners . . . . .	284
10.3	Incomplete LU Factorization Preconditioners . . . . .	287
10.3.1	ILU Factorizations . . . . .	288
10.3.2	Zero Fill-in ILU (ILU(0)) . . . . .	293
10.3.3	Level of Fill and ILU( $p$ ) . . . . .	296
10.3.4	Matrices with Regular Structure . . . . .	300
10.3.5	MILU . . . . .	305
10.4	Threshold Strategies and Incomplete LU with Threshold . . . . .	306
10.4.1	The ILUT Approach . . . . .	307
10.4.2	Analysis . . . . .	308
10.4.3	Implementation Details . . . . .	310
10.4.4	The ILUTP Approach . . . . .	312
10.4.5	The ILUS Approach . . . . .	314
10.4.6	The ILUC Approach . . . . .	316
10.5	Approximate Inverse Preconditioners . . . . .	320
10.5.1	Approximating the Inverse of a Sparse Matrix . . . . .	321
10.5.2	Global Iteration . . . . .	321
10.5.3	Column-Oriented Algorithms . . . . .	323
10.5.4	Theoretical Considerations . . . . .	324
10.5.5	Convergence of Self-Preconditioned MR . . . . .	326
10.5.6	AINVs via Bordering . . . . .	329
10.5.7	Factored Inverses via Orthogonalization: AINV . . . . .	331
10.5.8	Improving a Preconditioner . . . . .	333
10.6	Reordering for Incomplete LU . . . . .	333
10.6.1	Symmetric Permutations . . . . .	333
10.6.2	Nonsymmetric Reorderings . . . . .	335
10.7	Block Preconditioners . . . . .	337
10.7.1	Block Tridiagonal Matrices . . . . .	337
10.7.2	General Matrices . . . . .	339

10.8	Preconditioners for the Normal Equations . . . . .	339
10.8.1	Jacobi, SOR, and Variants . . . . .	340
10.8.2	IC(0) for the Normal Equations . . . . .	340
10.8.3	Incomplete Gram–Schmidt and ILQ . . . . .	342
	Exercises . . . . .	345
	Notes and References . . . . .	349
<b>11</b>	<b>Parallel Implementations</b>	<b>353</b>
11.1	Introduction . . . . .	353
11.2	Forms of Parallelism . . . . .	354
11.2.1	Multiple Functional Units . . . . .	354
11.2.2	Pipelining . . . . .	354
11.2.3	Vector Processors . . . . .	355
11.2.4	Multiprocessing and Distributed Computing . . . . .	355
11.3	Types of Parallel Architectures . . . . .	355
11.3.1	Shared Memory Computers . . . . .	356
11.3.2	Distributed Memory Architectures . . . . .	357
11.4	Types of Operations . . . . .	359
11.5	Matrix-by-Vector Products . . . . .	361
11.5.1	The CSR and CSC Formats . . . . .	362
11.5.2	Matvecs in the Diagonal Format . . . . .	364
11.5.3	The Ellpack-Itpack Format . . . . .	364
11.5.4	The JAD Format . . . . .	365
11.5.5	The Case of Distributed Sparse Matrices . . . . .	366
11.6	Standard Preconditioning Operations . . . . .	369
11.6.1	Parallelism in Forward Sweeps . . . . .	369
11.6.2	Level Scheduling: The Case of Five-Point Matrices . . . . .	370
11.6.3	Level Scheduling for Irregular Graphs . . . . .	370
	Exercises . . . . .	373
	Notes and References . . . . .	375
<b>12</b>	<b>Parallel Preconditioners</b>	<b>377</b>
12.1	Introduction . . . . .	377
12.2	Block Jacobi Preconditioners . . . . .	378
12.3	Polynomial Preconditioners . . . . .	379
12.3.1	Neumann Polynomials . . . . .	380
12.3.2	Chebyshev Polynomials . . . . .	381
12.3.3	Least-Squares Polynomials . . . . .	383
12.3.4	The Nonsymmetric Case . . . . .	386
12.4	Multicoloring . . . . .	389
12.4.1	Red-Black Ordering . . . . .	389
12.4.2	Solution of Red-Black Systems . . . . .	390
12.4.3	Multicoloring for General Sparse Matrices . . . . .	391
12.5	Multi-Elimination Incomplete LU . . . . .	392
12.5.1	Multi-Elimination . . . . .	393
12.5.2	ILUM . . . . .	394

12.6	Distributed Incomplete LU and Symmetric Successive Overrelaxation . . . . .	396
12.7	Other Techniques . . . . .	399
12.7.1	AINVs . . . . .	399
12.7.2	EBE Techniques . . . . .	399
12.7.3	Parallel Row Projection Preconditioners . . . . .	401
Exercises . . . . .		402
Notes and References . . . . .		404
<b>13</b>	<b>Multigrid Methods</b>	<b>407</b>
13.1	Introduction . . . . .	407
13.2	Matrices and Spectra of Model Problems . . . . .	408
13.2.1	The Richardson Iteration . . . . .	412
13.2.2	Weighted Jacobi Iteration . . . . .	414
13.2.3	Gauss–Seidel Iteration . . . . .	416
13.3	Intergrid Operations . . . . .	419
13.3.1	Prolongation . . . . .	419
13.3.2	Restriction . . . . .	421
13.4	Standard Multigrid Techniques . . . . .	422
13.4.1	Coarse Problems and Smoothers . . . . .	423
13.4.2	Two-Grid Cycles . . . . .	424
13.4.3	V-Cycles and W-Cycles . . . . .	426
13.4.4	FMG . . . . .	429
13.5	Analysis of the Two-Grid Cycle . . . . .	433
13.5.1	Two Important Subspaces . . . . .	433
13.5.2	Convergence Analysis . . . . .	435
13.6	Algebraic Multigrid . . . . .	437
13.6.1	Smoothness in AMG . . . . .	438
13.6.2	Interpolation in AMG . . . . .	439
13.6.3	Defining Coarse Spaces in AMG . . . . .	442
13.6.4	AMG via Multilevel ILU . . . . .	442
13.7	Multigrid versus Krylov Methods . . . . .	445
Exercises . . . . .		446
Notes and References . . . . .		449
<b>14</b>	<b>Domain Decomposition Methods</b>	<b>451</b>
14.1	Introduction . . . . .	451
14.1.1	Notation . . . . .	452
14.1.2	Types of Partitionings . . . . .	453
14.1.3	Types of Techniques . . . . .	453
14.2	Direct Solution and the Schur Complement . . . . .	456
14.2.1	Block Gaussian Elimination . . . . .	456
14.2.2	Properties of the Schur Complement . . . . .	457
14.2.3	Schur Complement for Vertex-Based Partitionings . . . . .	458
14.2.4	Schur Complement for Finite Element Partitionings . . . . .	460
14.2.5	Schur Complement for the Model Problem . . . . .	463

---

14.3	Schwarz Alternating Procedures . . . . .	465
14.3.1	Multiplicative Schwarz Procedure . . . . .	465
14.3.2	Multiplicative Schwarz Preconditioning . . . . .	470
14.3.3	Additive Schwarz Procedure . . . . .	472
14.3.4	Convergence . . . . .	473
14.4	Schur Complement Approaches . . . . .	477
14.4.1	Induced Preconditioners . . . . .	478
14.4.2	Probing . . . . .	480
14.4.3	Preconditioning Vertex-Based Schur Complements . . . . .	480
14.5	Full Matrix Methods . . . . .	481
14.6	Graph Partitioning . . . . .	483
14.6.1	Basic Definitions . . . . .	484
14.6.2	Geometric Approach . . . . .	484
14.6.3	Spectral Techniques . . . . .	486
14.6.4	Graph Theory Techniques . . . . .	487
	Exercises . . . . .	491
	Notes and References . . . . .	492
	<b>Bibliography</b>	<b>495</b>
	<b>Index</b>	<b>517</b>