✍1 Consider

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues of $A$? their algebraic multiplicities? their geometric multiplicities? Is one a semi-simple eigenvalue?

**Solution:** The eigenvalues of $A$ are 1, and 2. The algebraic multiplicity of 1 is 2. To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} u = 0.$$

There is only one solution vector (up to a product by a scalar) namely:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the geometric multiplicity is one. □

✍2 Same questions if $a_{33}$ is replaced by one.

**Solution:** The matrix become

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and now we have one eigenvalue algebraic multiplicity 3.

To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} u = 0.$$

we still get a geometric mult. of 1. $\square$

✍3 Same questions if in addition $a_{12}$ is replaced by zero.

**Solution: Solution:** The matrix become

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and we also have one eigenvalue with algebraic multiplicity 3. The geometric multiplicity increases to 2. $\square$

✍4 Show that there is at least one eigenvalue and eigenvector of $A$: $Ax = \lambda x$, with $\|x\|_2 = 1$

**Solution:** This comes from the fact that the equation $P_A(\lambda) = \det(A - \lambda I) = 0$ is a polynomial equation and as such it must have at least one root - a well-known result. $\square$

✍5 There is a unitary transformation $P$ such that $Px = e_1$. How do you define $P$?

**Solution:** This is just the Householder transform.. See Lecture notes set number 8. $\square$

✍6 Show that $PAP^H = \left(\begin{array}{c|c} \lambda & ** \\ \hline 0 & A_2 \end{array}\right)$.

**Solution:** This is equivalent to showing that $PAP^He_1 = \lambda e_1$. We have

$$PAP^He_1 = PAPe_1 = P(Ax) = P(\lambda x) = \lambda Px = \lambda e_1$$

$\square$

✍9 Another proof altogether: use Jordan form of $A$ and QR factorization **Solution:** Jordan form:

$$A = XJX^{-1}$$

Let $X = QR_0$ then:

$$A = QR_0JR_0^{-1}Q^H \equiv QRQ^H \quad \text{with} \quad R = R_0JR_0^{-1}$$

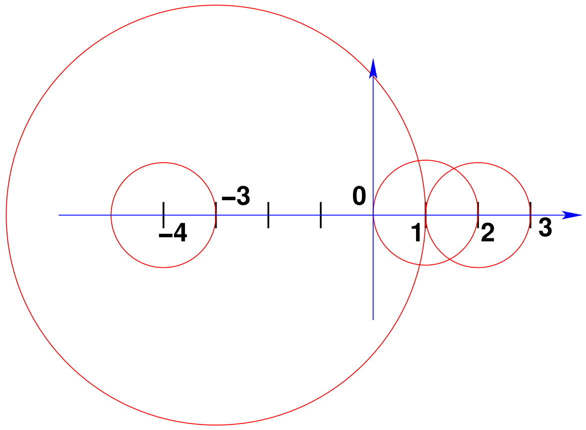
□

✍10 Find a region of the complex plane where the eigenvalues of the following matrix are located:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -2 & -3 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & -4 \end{pmatrix}$$

**Solution:** Use Gershgorin's theorem. There are 4 disks:

$$D_1 = D(1,1); \qquad D_2 = D(2,1)$$
$$D_3 = D(-3,4); \qquad D_4 = D(-4,1)$$




The last disk is included in the 3rd. The spectrum is included in the union of the 3 other disks. ☐

**Additional notes.**

In discussing Gerschgorin theorem it was stated:

➤ Refinement: if disks are all disjoint then each of them contains one eigenvalue

Question: Why?

**Solution:**

Consider the matrix $A(t) = D + t(A - D)$ where $D$ is the diagonal of $A$. Note $A(0) = D$, $A(1) = A$. Consider the $n$ disks as $t$ varies from $t = 0$ to $t = 1$. When $t = 0$ each disk contains exactly one eigenvalue. As $t$ increases (in a continuous way) fom 0 to one – each disk will still contain one eigenvalue - by a continuity argument [you cannot have an eigenvalue jump suddently - from one disk to another- this would be a dicontinuous behavior]. The argument can be adapted to the case where two disks touch each other at one point (only): it is now possible to have two eigenvalues at the intersection of the disks - coming from each of the t2o disks.