✍2 If $A \in \mathbb{R}^{m\times n}$ what are the dimensions of $A^\dagger$?, $A^\dagger A$?, $AA^\dagger$?

Solution: The dimension of $A^\dagger A$ is $n \times m$ and so $A^\dagger A$? is of size $n \times n$. Similarly, $AA^\dagger$ is of size $m \times m$. ☐

✍3 Show that $A^\dagger A$ is an orthogonal projector. What are its range and null-space?

Solution: One way to do this is to use the rank-one expansion: $A = \sum \sigma_i u_i v_i^T$. Then $A^\dagger = \sum \frac{1}{\sigma_i} v_i u_i^T$ and therefore,

$$A^\dagger A = \left[\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T\right] \times \left[\sum_{j=1}^{r} \sigma_j u_j v_j^T\right] = \sum_{j=1}^{r} v_j v_j^T$$

which is a projector. ☐

✍4 Same question for $AA^\dagger$ ..

Solution: In this case we have

$$AA^\dagger = \left[\sum_{j=1}^{r} \sigma_j u_j v_j^T\right] \left[\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T\right] \times = \sum_{j=1}^{r} u_j u_j^T$$

which is an orthogonal projector. ☐

✍5 Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

- Compute the singular value decomposition of A

Solution: The nonzero singular values of A are the square roots of the eigenvalues of

$$AA^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

These eigenvalues are 5 ± 4 and so $\sigma_1 = 3, \sigma_2 = 1$.

The matrix U of the left singular vectors is the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

If $A = U\Sigma V^T$, then $U' * A = \Sigma V^T$. Therefore to get V we use the relation: $V^T = \Sigma^{-1} * U' * A$. We have

$$U' * A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 4 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 0 & 4/3 & -1/3 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

□

- Find the matrix B of rank 1 which is the closest to A in 2-norm sense.

Solution: This is obtained by setting σ_2 to zero in the SVD - or - equivalently as $B = \sigma_1 u_1 v_1^T$. You will find

$$B = \begin{pmatrix} 1/2 & 0 & 2 & -1/2 \\ -1/2 & 0 & -2 & 1/2 \end{pmatrix}$$

$\square$

✍6 Show that r_ϵ equals the number sing. values that are $>\epsilon$

Solution: This result is based on the following easy-to-prove extension of the Young=Eckhart theorem:

$$\min_{rank(B) \le k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

which implies that if $\|A - B\|_2 < \sigma_{k+1}$ then $rank(B)$ must be $> k$ - or equivalently:

$$\|A - B\|_2 < \sigma_k \rightarrow rank(B) \ge k.$$

Let k be the number that satisfies $\sigma_{k+1} \le \epsilon < \sigma_k$ – which is the number of sing. values that are $> \epsilon$. Then we see from the above that $\|A - B\|_2 \le \epsilon$ implies that $rank(B) \ge k$. The smallest possible rank for B is precisely the integer k defined above. $\square$