Supplemental Document for "ADMM $\supseteq$ Projective Dynamics: Fast Simulation of Hyperelastic Models with Dynamic Constraints"

Matthew Overby, George E. Brown, Jie Li, and Rahul Narain

Appendix A
Convergence of ADMM on a quadratic problem

We consider the problem

$$\begin{aligned}\min_{\mathbf{x},\mathbf{z}}\ & f(\mathbf{x})+g(\mathbf{z})\\ \text{s.t. } & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{z}=\mathbf{c}.\end{aligned} \tag{1}$$

where f and g are quadratic functions with positive definite Hessians $\mathbf{H}_f$ and $\mathbf{H}_g$ respectively. Without loss of generality, we may translate $\mathbf{x}$ and $\mathbf{z}$ so that the minima of f and g occur at $\mathbf{x}=\mathbf{0}$ and $\mathbf{z}=\mathbf{0}$ respectively. Thus we have $f(\mathbf{x})=\frac{1}{2}\mathbf{x}^T\mathbf{H}_f\mathbf{x}$ and $g(\mathbf{z})=\frac{1}{2}\mathbf{z}^T\mathbf{H}_g\mathbf{z}$. We assume that the constant term $\mathbf{c}$ in the constraint is zero; a nonzero $\mathbf{c}$ does not affect the convergence rate, though of course it must lie in the image of $\mathbf{A}$ and $\mathbf{B}$. Finally, we rescale the variables via $\bar{\mathbf{x}}=\mathbf{P}\mathbf{x}$ and $\bar{\mathbf{z}}=\mathbf{Q}\mathbf{z}$ such that $\mathbf{P}^T\mathbf{P}=\mathbf{H}_f$ and $\mathbf{Q}^T\mathbf{Q}=\mathbf{H}_g$, and rescale the constraint by a matrix $\mathbf{W}$. This yields the equivalent problem

$$\begin{aligned}\min_{\bar{\mathbf{x}},\bar{\mathbf{z}}}\ & \frac{1}{2}\|\bar{\mathbf{x}}\|^2+\frac{1}{2}\|\bar{\mathbf{z}}\|^2\\ \text{s.t. } & \underbrace{\mathbf{W}\mathbf{A}\mathbf{P}^{-1}}_{\bar{\mathbf{A}}}\bar{\mathbf{x}}+\underbrace{\mathbf{W}\mathbf{B}\mathbf{Q}^{-1}}_{\bar{\mathbf{B}}}\bar{\mathbf{z}}=\mathbf{0}.\end{aligned} \tag{2}$$

Note that ADMM applied this problem has exactly the same convergence as when applied to the original objective $\min f(\mathbf{x})+g(\mathbf{x})$ with the rescaled constraint $\mathbf{W}\mathbf{A}\mathbf{x}+\mathbf{W}\mathbf{B}\mathbf{z}=\mathbf{0}$.

In the $\mathbf{x}$-step of ADMM, $\bar{\mathbf{x}}^{n+1}$ is determined completely by $\bar{\mathbf{z}}^n$ and $\mathbf{u}^n$ via

$$\begin{aligned}\bar{\mathbf{x}}^{n+1}&=\arg\min_{\bar{\mathbf{x}}}\Big(\frac{1}{2}\|\bar{\mathbf{x}}\|^2+\frac{\rho}{2}\|\bar{\mathbf{A}}\bar{\mathbf{x}}+\bar{\mathbf{B}}\bar{\mathbf{z}}^n+\mathbf{u}^n\|^2\Big)\\ &=-(\mathbf{I}+\rho\bar{\mathbf{A}}^T\bar{\mathbf{A}})^{-1}\bar{\mathbf{A}}^T(\bar{\mathbf{B}}\bar{\mathbf{z}}^n+\mathbf{u}^n).\end{aligned} \tag{3}$$

Therefore, the progress of the ADMM iterations is determined by the $\mathbf{z}$- and $\mathbf{u}$-steps. After some algebra, we obtain

$$\bar{\mathbf{z}}^{n+1}=\mathbf{S}\bar{\mathbf{B}}^T(\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}}\bar{\mathbf{z}}^n-(\mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T)\mathbf{u}^n), \tag{4}$$

$$\mathbf{u}^{n+1}=(\mathbf{I}-\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T)\big(-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}}\bar{\mathbf{z}}^n+(\mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T)\mathbf{u}^n\big), \tag{5}$$

where $\mathbf{R}=(\mathbf{I}+\rho\bar{\mathbf{A}}^T\bar{\mathbf{A}})^{-1}$ and $\mathbf{S}=(\mathbf{I}+\rho\bar{\mathbf{B}}^T\bar{\mathbf{B}})^{-1}$. This is a linear recurrence

$$\begin{bmatrix}\bar{\mathbf{z}}^{n+1}\\ \mathbf{u}^{n+1}\end{bmatrix}=\begin{bmatrix}\mathbf{S}\bar{\mathbf{B}}^T\\ \mathbf{I}-\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T\end{bmatrix}\begin{bmatrix}\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}} & \mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\end{bmatrix}\begin{bmatrix}\bar{\mathbf{z}}^{n}\\ \mathbf{u}^{n}\end{bmatrix}, \tag{6}$$

and its convergence rate is determined by the spectral radius of the recurrence matrix,

$$\begin{aligned}&r\left(\begin{bmatrix}\mathbf{S}\bar{\mathbf{B}}^T\\ \mathbf{I}-\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T\end{bmatrix}\begin{bmatrix}\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}} & \mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\end{bmatrix}\right)\\ &\quad=r\left(\begin{bmatrix}\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}} & \mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\end{bmatrix}\begin{bmatrix}\mathbf{S}\bar{\mathbf{B}}^T\\ \mathbf{I}-\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T\end{bmatrix}\right)\\ &\quad=r\big(\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T+(\mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T)(\mathbf{I}-\bar{\mathbf{B}}\mathbf{S}\bar{\mathbf{B}}^T)\big).\end{aligned} \tag{7}$$

In general, this expression cannot be simplified further. However, if $\rho=1$ and $\bar{\mathbf{B}}=\mathbf{I}$, then we obtain $\mathbf{S}=(\mathbf{I}+\rho\bar{\mathbf{B}}^T\bar{\mathbf{B}})^{-1}=\frac{1}{2}\mathbf{I}$, and the convergence rate becomes simply

$$r\Big(\frac{1}{2}\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T+\frac{1}{2}(\mathbf{I}-\bar{\mathbf{A}}\mathbf{R}\bar{\mathbf{A}}^T)\Big)=\frac{1}{2}. \tag{8}$$

This is achieved when $\bar{\mathbf{B}}=\mathbf{W}\mathbf{B}\mathbf{Q}^{-1}=\mathbf{I}$, i.e. $\mathbf{Q}=\mathbf{W}\mathbf{B}$. Further, as $\mathbf{Q}$ is any matrix which satisfies $\mathbf{Q}^T\mathbf{Q}=\mathbf{H}_g$, we only require $(\mathbf{W}\mathbf{B})^T(\mathbf{W}\mathbf{B})=\mathbf{H}_g$, equivalently, $\frac{1}{2}\|\mathbf{W}\mathbf{B}\mathbf{x}\|^2=g(\mathbf{x})$.

Appendix B
Proof that projective dynamics $\approx$ ADMM

We apply ADMM to the projective dynamics energy

$$U_i(\mathbf{z}_i)=\min_{\mathbf{p}_i\in\mathcal{C}_i}\frac{k_i}{2}\|\mathbf{z}_i-\mathbf{p}_i\|^2. \tag{9}$$

In our formulation of ADMM, we have one parameter $\mathbf{W}$. We define $\mathbf{W}$ via $\mathbf{W}_i=w_i\mathbf{I}=\sqrt{k_i}\mathbf{I}$, so that $\mathbf{W}^T\mathbf{W}=\mathbf{K}$. Then the energy can be conveniently expressed in terms of a single constraint manifold, $\mathcal{C}=\mathcal{C}_1\times\mathcal{C}_2\times\cdots\times\mathcal{C}_m$:

$$\begin{aligned}U_*(\mathbf{z})&=\min_{\mathbf{p}\in\mathcal{C}}\frac{1}{2}(\mathbf{z}-\mathbf{p})^T\mathbf{K}(\mathbf{z}-\mathbf{p}) &(10)\\ &=\min_{\mathbf{p}\in\mathcal{C}}\frac{1}{2}\|\mathbf{W}(\mathbf{z}-\mathbf{p})\|^2. &(11)\end{aligned}$$

• *M. Overby, G.E. Brown, J. Li, and R. Narain are with the University of Minnesota.*

Now the $\mathbf{z}$-step of ADMM becomes

$$\mathbf{z}^{n+1} = \arg\min_{\mathbf{z}} \Big(U_*(\mathbf{z}) + \frac{1}{2}\|\mathbf{W}(\mathbf{D}\mathbf{x}^{n+1} - \mathbf{z} + \bar{\mathbf{u}}^n)\|^2 \Big) \tag{12}$$

$$= \arg\min_{\mathbf{z}} \left(\min_{\mathbf{p}\in\mathcal{C}} \frac{1}{2}\|\mathbf{W}(\mathbf{z}-\mathbf{p})\|^2 + \frac{1}{2}\|\mathbf{W}(\mathbf{z}-\mathbf{y})\|^2 \right) \tag{13}$$

where $\mathbf{y} = \mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n$. Consider the underlying minimization

$$\min_{\mathbf{z},\mathbf{p}\in\mathcal{C}} \frac{1}{2}\|\mathbf{W}(\mathbf{z}-\mathbf{p})\|^2 + \frac{1}{2}\|\mathbf{W}(\mathbf{z}-\mathbf{y})\|^2. \tag{14}$$

For any fixed $\mathbf{p} \in \mathcal{C}$, the minimum is attained at $\mathbf{z} = \frac{1}{2}(\mathbf{p}+\mathbf{y})$ and its value is $\frac{1}{4}\|\mathbf{W}(\mathbf{p}-\mathbf{y})\|^2$. Therefore, the optimal $\mathbf{p}$ must minimize $\|\mathbf{W}(\mathbf{p}-\mathbf{y})\|^2$. For our choice of $\mathbf{W}$ and $\mathcal{C}$ this amounts to minimizing $w_i\|\mathbf{p}_i - \mathbf{y}_i\|^2$ independently for each i, that is, choosing $\mathbf{p}_i = \mathrm{proj}_{\mathcal{C}_i}\mathbf{y}_i = \mathrm{proj}_{\mathcal{C}_i}(\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n)$. So in fact we have

$$\mathbf{p}^{n+1} = \mathrm{proj}_{\mathcal{C}}(\mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n), \tag{15}$$

$$\mathbf{z}^{n+1} = \frac{1}{2}\left(\mathbf{p}^{n+1} + \mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n\right). \tag{16}$$

Armed with (15)–(16), we will now eliminate $\mathbf{z}$ from the ADMM update rules in favour of $\mathbf{p}$. The $\mathbf{u}$-update becomes

$$\bar{\mathbf{u}}^{n+1} = \bar{\mathbf{u}}^n + \mathbf{D}\mathbf{x}^{n+1} - \mathbf{z}^{n+1} \tag{17}$$

$$= \bar{\mathbf{u}}^n + \mathbf{D}\mathbf{x}^{n+1} - \frac{1}{2}\left(\mathbf{p}^{n+1} + \mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n\right) \tag{18}$$

$$= \frac{1}{2}\left(\mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n - \mathbf{p}^{n+1}\right). \tag{19}$$

Conveniently, this also means that after the $\bar{\mathbf{u}}$-update,

$$\mathbf{z}^{n+1} - \bar{\mathbf{u}}^{n+1} = \frac{1}{2}\left(\mathbf{p}^{n+1} + \mathbf{D}\mathbf{x}^{n+1} + \mathbf{u}^n\right) - \frac{1}{2}\left(\mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n - \mathbf{p}^{n+1}\right) \tag{20}$$

$$= \mathbf{p}^{n+1}. \tag{21}$$

The $\mathbf{x}$-update is now

$$\mathbf{x}^{n+1} = \left(\mathbf{M} + \Delta t^2\mathbf{D}^T\mathbf{W}^T\mathbf{W}\mathbf{D}\right)^{-1} \left(\mathbf{M}\tilde{\mathbf{x}} + \Delta t^2\mathbf{D}^T\mathbf{W}^T\mathbf{W}(\mathbf{z}^n - \bar{\mathbf{u}}^n)\right) \tag{22}$$

$$= \left(\mathbf{M} + \Delta t^2\mathbf{D}^T\mathbf{W}^T\mathbf{W}\mathbf{D}\right)^{-1}\left(\mathbf{M}\tilde{\mathbf{x}} + \Delta t^2\mathbf{D}^T\mathbf{W}^T\mathbf{W}\mathbf{p}^n\right), \tag{23}$$

exactly like the $\mathbf{x}$-update in projective dynamics. Instead of the $\mathbf{z}$-update we have

$$\mathbf{p}^{n+1} = \mathrm{proj}_{\mathcal{C}}(\mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n) \tag{24}$$

$$\iff \mathbf{p}_i^{n+1} = \mathrm{proj}_{\mathcal{C}_i}(\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}_i^n), \tag{25}$$

which is almost exactly like the $\mathbf{p}$-update in projective dynamics, except for the presence of the dual variables $\bar{\mathbf{u}}_i$. Finally, the $\mathbf{u}$-update remains

$$\bar{\mathbf{u}}^{n+1} = \frac{1}{2}\left(\mathbf{D}\mathbf{x}^{n+1} + \bar{\mathbf{u}}^n - \mathbf{p}^{n+1}\right) \tag{26}$$

$$\iff \bar{\mathbf{u}}_i^{n+1} = \frac{1}{2}\left(\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}_i^n - \mathbf{p}_i^{n+1}\right) \tag{27}$$

which has no counterpart in projective dynamics.

So far we have seen that for a general constraint manifolds $\mathcal{C}_i$, projective dynamics and ADMM are extremely similar, with the only difference being the presence of the $\bar{\mathbf{u}}_i$ variables and their corresponding update rules. In the special case when the constraints are *linear*, that is, the manifolds $\mathcal{C}_i$ are affine, we further show that the two algorithms become identical.

Let $\mathcal{C}_i$ be an affine subspace with normal space $\mathcal{N}_i$. Then the projection operator $\mathrm{proj}_{\mathcal{C}_i}$ has the properties that

$$\mathbf{z}_i - \mathrm{proj}_{\mathcal{C}_i}\mathbf{z}_i \in \mathcal{N}_i, \tag{28}$$

$$\forall \mathbf{n} \in \mathcal{N}_i : \mathrm{proj}_{\mathcal{C}_i}(\mathbf{z}_i + \mathbf{n}) = \mathrm{proj}_{\mathcal{C}_i}\mathbf{z}_i. \tag{29}$$

We can see that

$$\bar{\mathbf{u}}_i^{n+1} = \frac{1}{2}\left(\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}_i^n - \mathbf{p}_i^{n+1}\right) \tag{30}$$

$$= \frac{1}{2}\left((\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}_i^n) - \mathrm{proj}_{\mathcal{C}}(\mathbf{D}_i\mathbf{x}^{n+1} + \bar{\mathbf{u}}_i^n)\right) \tag{31}$$

$$\in \mathcal{N}_i, \tag{32}$$

and so

$$\mathbf{p}_i^{n+1} = \mathrm{proj}_{\mathcal{C}_i}\left(\mathbf{D}_i\mathbf{x}^{n+1} + \mathbf{u}_i^n\right) \tag{33}$$

$$= \mathrm{proj}_{\mathcal{C}_i}\mathbf{D}_i\mathbf{x}^{n+1} \tag{34}$$

as long as $\mathbf{u}_i^0 \in \mathcal{N}_i$ (for example, if we initialize $\mathbf{u}_i^0 = \mathbf{0}$).

This proves the equivalence of projective dynamics and ADMM for linear constraints. Furthermore, nonlinear constraints that are smooth can be well approximated by a linearization in the neighborhood of the solution, so both algorithms should behave similarly as they approach convergence.