How to 'Glue' a Robust Smart-Grid?

A "Finite-Network" Theory for Interdependent Network Robustness

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ABSTRACT

Smart-grids are made up of two interdependent constituents: a power production and distribution network catering to a geographic area and a communication network that helps regulate and control the power-grid from which it derives its electricity. Thus, failures in one of the two networks can lead to failures in the other and in some cases cause a cascade or domino effect, bringing down the entire system completely. In this work, we present a theoretical framework to model and study the structural properties of such interdependent networks based on a topological interpretation [4] of the Moore-Penrose pseudo-inverse of the graph Laplacians ($\mathbf{L}^+$). Using this framework, we study how the way in which node pairs in two networks are coupled or "glued" together (thereby introducing interdependence) affects the overall robustness of the resulting *interdependent* networks. Our study leads to some surprising (and somewhat counter-intuitive) results.

1. INTRODUCTION

Modern infrastructure networks are becoming increasingly complex and dependent on one another. An example of such an interdependence is that of an electrical power-grid network regulated by a communication network which in turn depends on the same power-grid for its electrical supply. Due to such interdependence, failures of elements in one network, e.g., a small fraction of nodes in a communication network that is used to control and communicate elements in a smart grid, can induce failures in the other, i.e. the power grid network, which would in turn cause further failures in the communication and control network, thus producing a cascade of failures in the interdependent networks. In the recent past, electrical blackouts, like the one in Italy on 28 September 2003 [5], have in fact been caused by such cascaded failures.

Clearly, the extent to which *random* failures or targeted attacks can lead to a cascaded failure, of course, depends on the structural properties of the constituent networks. In their seminal work, Buldyrev et al [2] have demonstrated that interdependent networks can behave very differently from each of their constituents. In particular, two robust *power-law* networks when made interdependent via "random coupling" may become more vulnerable to random failures. Their work, and those of others, quantify the structural robustness of interdependent networks in terms of asymptotic statistical properties such as the existence of *giant connected components* under random failures. As in the case of robustness of single networks, while this complex network theory characterization of network robustness provides valuable insight into the *general statistical properties of interdependences of classes of random graphs/networks*, they are not very useful in practice, as real networks are *deterministic* and *finite*. In particular, *engineered* infrastructure networks such as power-grids and communication networks, are designed to perform certain specific functions. There topological structures reflect and are constrained by the functional roles of various nodes as well as their geographical locations that are dictated by, say, user population or other resources. In other words, their structures may differ significantly from (theoretically generated) random networks, and the *interdependencies* between two networks (e.g., communication networks and power grids are not arbitrary, but often are determined by geographical and other constraints).

In this work, we propose a theoretical framework for assessing structural robustness of interdependent networks which is not dependent on specific assumptions of structure like a power-law degree distribution. In doing so, we address the following important questions related to vulnerability assessment and protection of interdependent networks: (a) how can we characterize the robustness of the interdependent network on a whole? (b) how is the overall robustness of two interdependent networks affected by the manner in which the two networks are *coupled* (or "glued") together? (c) how do we judicially select an appropriate *coupling* function, namely, select an appropriate collection of *coupled* node pairs to introduce "interdependence" (i.e., where the two networks are "glued" together) so that the resulting interdependent networks are more robust to random failures or targeted attacks? The answer to the last question provides insights as to how we can harden two interdependent networks.

In the following we demonstrate that these questions can be answered – at least theoretically – by studying the topological properties captured by our "geometry of networks" approach [3], namely, by using the Moore-Penrose pseudo-inverse of the graph Laplacians ($\mathbf{L}^+$) for the individual networks and that of the interdependent network. Based on

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the topological interpretations of $\mathbf{L}^+$ [4], in particular, the structural centrality metrics and Kirchhoff index, we develop a (*deterministic*) "finite-network" theory to study the *robustness of interdependence*. This theory enables us to mathematically quantify the structural centrality and roles of *coupled* nodes (as well as uncoupled nodes) in the interdependent networks as well as the robustness of interdependent networks as a whole. More importantly, this theory allows us to explicitly study how the way node pairs in two networks are *coupled* or "glued" together (thereby introducing interdependence) affects the overall robustness of the resulting interdependent networks. Our study leads to some *surprising* (and somewhat counter-intuitive) results: i) simply "gluing" together of structurally most central node pairs in the two constituent networks (when considered independently) does not always result in the most structurally robust *interdependent network* ii) coupling a *large* number of structurally least central node pairs in the two constituent networks often leads to more robust interdependent networks than coupling the same number of structurally most central node pairs. Intuitively, this result suggests that by *diffusing and distributing inter-dependencies among a large number of (geographically dispersed) node pairs in the two constituent networks produce more robust interdependent networks.*

2. GEOMETRY OF NETWORKS AND NETWORK ROBUSTNESS

In this section, we provide a brief overview of our "geometry of networks" approach [3], namely, via the $\mathbf{L}^+$ embedding, for studying the robustness of a *single* network. In particular, we present the *topological* interpretations of $\mathbf{L}^+$ with respect to the bi-partitions of a network (see [4] for details).

We model our network N as a simple graph $G(V, E)$. Let $|V(G)| = n$ be the number of vertices/nodes, henceforth called the order of the graph, and $|E(G)| = m$ be the number of edges/links in G. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the adjacency matrix of $G(V, E)$, with elements $a_{ij} = a_{ji} = 1$ if $e(i, j) \in E(G)$ and $a_{ij} = a_{ji} = 0$ otherwise. Let $\mathbf{D}$ be a diagonal matrix with $d(i)$ on the i^{th} diagonal, where $d(i)$ is the degree of node i. The (unnormalized) graph Laplacian $\mathbf{L}$ is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$. $\mathbf{L}$ is symmetric, doubly-centered and positive semi-definite. Let $\mathbf{L}^+ = [l^+_{ij}]$ denote the Moore-Penrose pseudo inverse [1] of $\mathbf{L}$. $\mathbf{L}^+$ is also symmetric, doubly-centered and positive semi-definite, and admits an eigen-decomposition $\mathbf{L}^+ = \Phi\Lambda^+\Phi'$, where Φ is the matrix whose columns are the mutually perpendicular eigen-vectors of $\mathbf{L}^+$ and Λ^+ is a diagonal matrix with the eigen-values of $\mathbf{L}^+$ (i.e. the reciprocals of the non-zero eigen-values of $\mathbf{L}$). Moreover, $\mathbf{L}^+ = \Phi\Lambda^+\Phi' = \mathbf{X}'\mathbf{X}$, where the i^{th} column of the matrix $\mathbf{X}$ represents the coordinates for node i in an n-dimensional Euclidean space. The squared-length of the position vector, i.e., the distance of node i from the origin, is given by l^+_{ii}.

In [3], we define the *structural centrality* of node i as $\mathcal{C}^*(i) := 1/l^+_{ii}$. The *Kirchoff index* of a network is therefore given by $\mathcal{K}(G) = \sum_{i=1}^n l^+_{ii} = \sum_{i=1}^n 1/\mathcal{C}^*(i)$. *Geometrically*, we see that closer a node i is to the origin, the higher its structural centrality $\mathcal{C}^*(i)$ is. Likewise, the smaller $\mathcal{K}(G)$ is, the more compact the embedding of the network is.

Previously in [3], we have provided two interpretations for $\mathcal{C}^*(i) := 1/l^+_{ii}$, in terms of random walks and electrical networks, and demonstrated that $\mathcal{C}^*(i)$ can be used to measure the structural roles of nodes in a network, whereas $\mathcal{K}(G)$ can be used to measure the overall robustness of a network. In the following, we provide yet another interpretation – a *topological* interpretation – of $\mathbf{L}^+$ in terms of (connected) bi-partitions and spanning (sub-)trees.

DEFINITION 1. *A (connected)* bi-partition $P = (S, S')$ *of a graph* G *consists of two sub-graphs,* S *and* S'*, where* $V(S) \cap V(S') = \phi$ *and* $V(S) \cup V(S') = V(G)$*, and each sub-graph* S *(resp.* S'*) is connected.*

Let $E(S, S')$ denote the sets of "cross-edges" of the partition $P = (S, S')$, i.e., $E(S, S') := \{(u, v) \in E, u \in S, v \in S'\}$. We see that a partition $P = (S, S')$ represents a state of the network in which $E(S, S')$ edges have failed, leaving the network partitioned into two mutually exclusive sub-networks. It is this view that provides a significant insight into the true nature of structural centrality and Kirchoff index.

Let $\mathcal{P}(G)$ denote the set of all (connected) bi-partitions of the graph, and given a bi-partition $P = (S, S') \in \mathcal{P}(G)$, let $\mathcal{T}_S$ and $\mathcal{T}_{S'}$ represent the sets of spanning trees defined over the nodes of S and S', respectively (since S and S' are connected subgraphs of G, both $\mathcal{T}_S$ and $\mathcal{T}_{S'}$ are non-empty). It can be shown [4] that

$$l^+_{ii} \propto \frac{\sum_{P\in\mathcal{P}(G), i\in V(S)} |\mathcal{T}(S)||\mathcal{T}(S')||V(S')|}{\sum_{P\in\mathcal{P}(G)} |\mathcal{T}(S)||\mathcal{T}(S')||E(S, S')|} \quad (1)$$

and

$$\mathcal{K}(G) = \frac{n-1}{n} \frac{\sum_{P\in\mathcal{P}(G)} |\mathcal{T}(S)||\mathcal{T}(S')||V(S)||V(S')|}{\sum_{P\in\mathcal{P}(G)} |\mathcal{T}(S)||\mathcal{T}(S')||E(S, S')|} \quad (2)$$

From eq.(1), it is not too hard to see that if $l^+_{ii} < l^+_{jj}$ (thus $\mathcal{C}^*(i) > \mathcal{C}^*(j)$), then "on the average" node i is more likely to lie in the larger of the remaining two sub-networks and node j lies in the smaller of the two after a network partition. Hence node i is in generally better connected (or structurally more central) than node j. Similarly, from eq.(2), we see that smaller $\mathcal{K}(G)$ implies that in general more edges (thus larger $|E(S, S')|$) have to be removed to partition G into two connected sub-networks.

3. MODELING INTERDEPENDENT NETWORKS

In this section we briefly describe the basic notations and a simple graph model for interdependent networks. In particular, we introduce a *coupling function*, $\mathcal{C} = \{[u, v]$, which specifies how and where two constituent networks are *coupled* or "glued" together to introduce inter-dependencies among the two networks and form a single *interdependent* network.

Given two networks, N_1 and N_2, we represent them in terms of their respective graphs: $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. Also, for the sake of simplicity, we assume that $|V_1| \approx |V_2|$, i.e., the two networks are of comparable sizes. Let $\mathcal{C}$ denote a collection of node pairs $[u, v]$, one from each constituent network, i.e., $\mathcal{C} = \{[u, v], u \in V_1, v \in V_2\}$. We refer to $\mathcal{C}$ as a *coupling function*, and each node pair $[u, v]$ in $\mathcal{C}$ a *coupled* node pair. Intuitively, the coupled node pairs are where an inter-dependency between two constituent networks are

introduced. The cardinality of $\mathcal{C}$, $k = |\mathcal{C}|$, represents the number of inter-dependencies, i.e., the number of *coupled* node pairs. Hence given a coupling function $\mathcal{C}$, two networks, N_1 and N_2, form as a whole a single *interdependent* network, denoted as $G_c(V_c, E_c)$.

As a graph, the interdependent network $G_c(V_c, E_c)$ can be defined as follows. Let $[u, v], u \in V_1$ and $v \in V_2$, be a coupled node pair in $\mathcal{C}$. When glued/coupled together, u and v will result in a new node $u \otimes v$ in G_c such that each edge $e(u, x) \in E_1$ will now create an edge $e(u \otimes v, x) \in E(G_c)$, if x is an uncoupled node in N_1; it will create an edge $e(u \otimes v, x \otimes y) \in E(G_c)$ if x is also a node in another coupled pair $[x, y] \in \mathcal{C}$. Similarly, each edge $e(v, y) \in E_2$ will create an edge $e(u \otimes v, y) \in E(G_c)$, if y is an uncoupled node in N_2; it will create an edge $e(u \otimes v, x \otimes y) \in E(G_c)$ if y is also a node in another coupled pair $[x, y] \in \mathcal{C}$. In other words, where there were two vertices u and v in the individual networks, we create a macro-vertex $u \otimes v$ in the glued network with a neighbor set that is a union of the neighbors of u and v in the original networks. This representation clearly captures the interdependent nature of the two vertices in question, whereby the macro-vertex $u \otimes v$ fails if either u or v fail in their individual networks. Similarly, the failure of edge $e(u, x)$ in G_1, results in the failure of $e(u \otimes v, x)$ in G_c. Uncoupled nodes and their associated edges (to other uncoupled nodes) in each of the two individual networks are transported to the coupled/interdependent $G_c(V_c, E_c)$ as is. Thus, if the number of couplings is κ, then $|V(G_c)| = |V(G_1)| + |V(G_2)| - \kappa$ and $|E(G_c)| = |E(G_1)| + |E(G_2)|$. So the order of the glued network reduces by κ as compared to the total of its constituents, but its volume (number of edges) is the same.

Given this definition of an interdependent network $N_c := G_c(V_c, E_c)$ formed by two constituent networks, N_1 and N_2, via the coupling function $\mathcal{C}$, we can directly apply the results in Section 2: using the Moore-Penrose pseudo-inverse of the graph Laplacians ($\mathbf{L}_c^+$) for the interdependent network $G_c(V_c, E_c)$, we define the corresponding *structural centrality* metrics and the *Kirchhoff index* for the interdependent network to measure the structural roles of individual (coupled and uncoupled) nodes as well as the overall robustness of the interdependent network.

4. EFFECT OF COUPLING FUNCTIONS ON NETWORK ROBUSTNESS

Of particular importance, and one of the principal contributions of our work, is that our theory enables us to investigate the effect of the coupling function $\mathcal{C}$ on the overall robustness of the resulting interdependent network. In other words, it allows us to vary the manner in which we select the node pairs from the two constituent networks to be glued together, and study the "optimal" way of introducing interdependencies so as enhance or "harden" the overall robustness of the resulting interdependent network.

For this purpose, we adopt a structural centrality based ordering of the nodes for the selection process. First, we rank the nodes in the networks N_1 and N_2 in terms of their structural centrality values in their respective networks. Having obtained the ranks, we then define the following three ways of gluing them together: (a) *high-high* i.e. the κ-highest ranked nodes from G_1 with the κ-highest ranked nodes from G_2, (b) *low-low* i.e. the κ-lowest ranked nodes from G_1 with

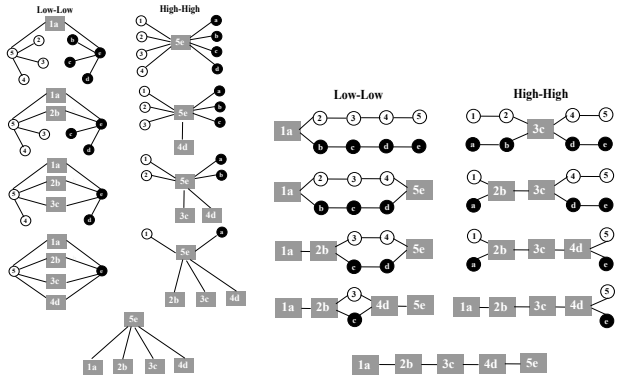


Figure 1: Coupling in stars and paths ($n = 5$).

the κ-lowest ranked nodes from G_2 and (c) *random* i.e. κ random pairs from G_1 and G_2. The overall robustness of the coupled/interdependent network is then measured in terms of its Kirchhoff index i.e. $\mathcal{K}(G_c)$. In the following, we first use a simple example (networks with tree topologies) and then a network with realistic network topology (the Italian power grid) to illustrate the coupling process and the effect of different coupling functions on the robustness of resulting interdependence networks.

4.1 When Both Networks are Trees

We consider a simple case to illustrate our theory, where two constituent networks are both trees. We study the coupling of two types of trees: stars and chains/paths of order n, which represent the most well connected (compact) and the least connected of all trees for any given n. More importantly, in the context of power-grids (and to an extent in communication networks), a star topology represents a production/distribution center i.e. the root of the star, while the pendants represent the consumers/first-hop *relay-ers*. Similarly, a chain represents a linear sub-network formed by a series of relay-ers to disseminate power or information. Using eq.(1), we observe that for a star topology:

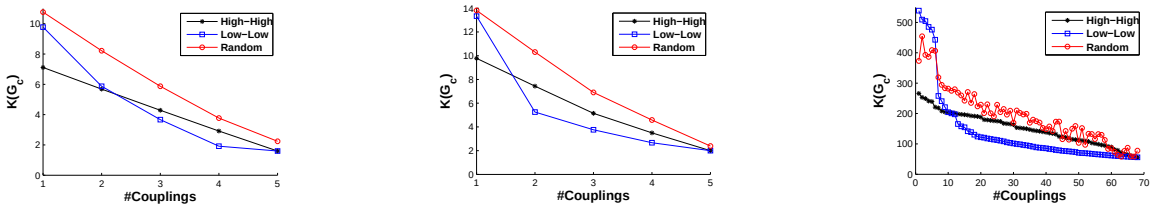
$$l^+_{root} = \frac{n-1}{n^2}, \quad and \quad l^+_{pendant} = \frac{n^2-n-1}{n^2} \tag{3}$$

It is easy to see that for $n > 2$, the root of the star has a lower l^+_{ii} value than the pendants and is therefore more structurally central. Similarly, for a chain of order n, the l^+_{ii} for the i^{th} vertex from the end is given as:

$$l^+_{ii} = \frac{6i^2 - 6(n+1)i + 2n^2 + 3n + 1}{6n} \tag{4}$$

The form in (4) is parabolic in l^+_{ii} and i. Clearly, for a given n, the minima is attained when $i = \lceil n/2 \rceil$, i.e. at the middle node/s of the chain, and the maxima is attained when $i = 1 = n$ i.e. at the pendants. Thus, the structural centrality of nodes in a chain decreases as we move from the center of the chain towards the pendants on either side.

Next, we demonstrate the $low-low$ and $high-high$ gluing strategies for a star and a chain respectively with another star and chain of the same orders in Fig. 1, for increasing values of $\kappa : 1 \leq \kappa \leq n$. Observe that for $\kappa \geq 2$, the $low - low$ strategy produces multiple cycles in the interdependent networks, thereby providing alternate connectivities between nodes and safeguarding against eventual edge



(a) Star (b) Path (c) Italian power grid

Figure 2: $\mathcal{K}(G_c)$ for glued/coupled networks for the three different coupling functions.

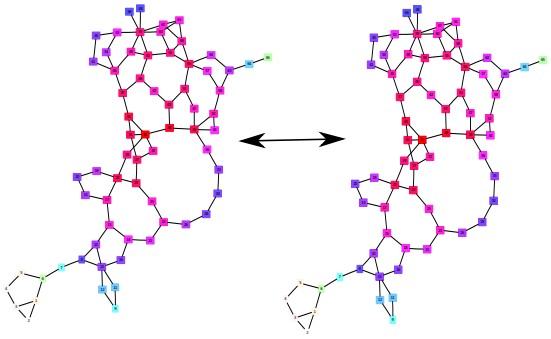


Figure 3: The Italian power grid network coupled with itself: *Red* → *Turquoise* decreasing structural centrality.

failures. It is well known that greater the number of cycles in a graph, higher the count of spanning trees which signifies better redundant connectivities between node pairs. In contrast, the $high - high$ strategy produces small loops between adjacent nodes and leaves the overall structure rather tree like. Thus the $low - low$ strategy results in more robust glued/coupled networks for $\kappa \geq 2$ in both the star and chain topologies.This effect is well captured in the values of $\mathcal{K}(G_c)$ shown for $1 \leq \kappa \leq n$ in Fig. 2(a) and (b), where the $low - low$ strategy attains lowest values of $\mathcal{K}(G_c)$ for $\kappa > 2$ (note that smaller $\mathcal{K}(G_c)$ is, more robust the network is). We see that when κ is small, pairing and coupling the most structurally central nodes (in the constituent networks) produces more robust networks than other coupling strategies. However, somewhat counter-intuitively, when κ increases and becomes sufficiently large, pairing and coupling structurally least central node pairs in the two constituent networks produces more robust interdependent networks. This result suggests that distributing interdependencies amongst (geographically) disparate nodes may result in more robust interdependent networks. We explore this further with the help of a real world power-grid network in the next section.

4.2 The Italian Power Grid Network Example

The Italian power grid network (see Fig. 3) is a network of order $n = 68$ and $m = 93$. The nodes in fig. 3 have been colored by their structural centralities. Notice how the structural centrality reduces as we move towards the periphery of the network. We now glue the Italian power grid network with an exact copy of itself, which represents a communication network (*cf.* [2]) using various coupling functions $\mathcal{C}$ for increasing values of κ. Once again, (see fig. 2(c)) we obtain the same surprising (and somewhat counter-intuitive) results that only for small κ, pairing and coupling the most structurally central nodes (in the constituent networks) produce more robust (i.e., smaller $\mathcal{K}(G_c)$) networks than other coupling strategies; for $\kappa > 10$, the $low - low$ strategy produces more robust interdependent network than other strategies. We note that as κ increases, the $low - low$ strategy glues peripheral nodes thereby creating longer cycles in the network as compared to the other two strategies. Such longer cycles safeguard network wide connectivities against random edge failures, thus resulting in more robust interdependent network structures. We also observe similar results for the western states power grid network in the US.

5. CONCLUSIONS

In this work, we presented a theoretical framework to assess structural properties of interdependent networks, based on the topological properties of the Moore-Penrose pseudoinverse of the graph Laplacians. We demonstrated that the structural centrality of nodes can be used to select nodes for gluing the two networks together and the robustness of the interdependent network can be measured in terms of its Kirchhoff index, given by the trace of the $\mathbf{L}^+$ of the interdependent network. With the help of example tree structures and the Italian power grid network, we presented comparative results for three gluing strategies based on the structural centralities of the nodes. Of the three strategies, the $low - low$ strategy eventually wins out, producing the most robust coupled structures (interdependent networks). Our results suggest that by *diffusing and distributing interdependencies among a large number of (geographically dispersed) node pairs in the two constituent networks produce more robust interdependent networks.*

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