Problem 1
Let \( P \) and \( R \) be relation symbols and let \( c \) be a constant symbol. Argue that the following are valid:

1. \( (\exists x)(P(x) \supset (\forall x)P(x)) \), and
2. \( (\exists y)(\forall x)R(x,y) \supset (\forall x)(\exists y)R(x,y) \).

Show that the following is not valid by describing an interpretation that does not satisfy it:
\( (\forall x)(\exists y)R(x,y) \supset (\exists y)(\forall x)R(x,y) \).

Problem 2
Assume a language that has equality and a two place predicate symbol \( P \). For each of the following conditions, find a sentence \( S \) such that the interpretation \( \mathcal{A} = \langle D, I \rangle \) is a model of \( S \) if and only if the condition is met.

1. \( D \) has exactly two members.
2. \( P^I \) is a function from \( D \) to \( D \).
3. \( P^I \) is a permutation of \( D \), i.e. \( P^I \) is a one-to-one function with domain and range equal to \( D \).

Problem 3
(Think of the next three problems together so that you are able to extract a useful pattern from one to apply to the next.) Write a sentence \( \Phi \) involving the two-place relation symbol \( R \) that has the following properties:

1. \( \Phi \) is not true in any interpretation with a one element domain, and
2. for each \( n \geq 2 \), \( \Phi \) is true in some interpretation whose domain has \( n \) elements.

Problem 4
Write a sentence \( \Phi \) involving the two-place relation symbol \( R \) that has the following properties:

1. \( \Phi \) is not true in any interpretation with a one or two element domain, and
2. for each \( n \geq 3 \), \( \Phi \) is true in some interpretation whose domain has \( n \) elements.
Problem 5

Write a sentence $\Phi$ involving the two-place relation symbol $R$ that has the following properties:

1. $\Phi$ is not true in any interpretation with a finite domain, and
2. there is some interpretation relative to each infinite domain that satisfies $\Phi$.

Problem 6

Let $A_1 = \langle D_1, I_1 \rangle$ and $A_2 = \langle D_2, I_2 \rangle$ be two interpretations for a language $L$. A function $h : D_1 \to D_2$ is a homomorphism of $A_1$ into $A_2$ if the following conditions are satisfied:

- For every constant symbol $c$ of $L$, it is the case that $h(c^{I_1}) = c^{I_2}$.
- For every $n$-ary function symbol $f$ and for every $\langle a_1, \ldots, a_n \rangle \in A^n$, it is the case that $h(f^{I_1}(a_1, \ldots, a_n)) = f^{I_2}(h(a_1), \ldots, h(a_n))$.
- For every $n$-ary relation symbol $R$ and for every $\langle a_1, \ldots, a_n \rangle \in A^n$, it is the case that $\langle a_1, \ldots, a_n \rangle \in R^{I_1}$ iff $\langle h(a_1), \ldots, h(a_n) \rangle \in R^{I_2}$.

The homomorphism $h$ is said to be from $A_1$ onto $A_2$ if the range of $h$ is all of $D_2$. Finally $h$ is said to be an isomorphism of $A_1$ into (onto) $A_2$ if it is a one-to-one function in addition to being a homomorphism of $A_1$ into (onto) $A_2$.

Given a homomorphism $h$ of $A_1$ into $A_2$ and an assignment $\phi$ in $A_1$, we can define an assignment $\phi_h$ in $A_2$ as follows: if $\phi$ maps a variable $x$ to $a$ in $A_1$, then $\phi_h$ maps $x$ to $h(a)$.

Let $A_1$ and $A_2$ be two interpretations and let $\phi$ be an assignment in $A_1$. Further let $h$ be a homomorphism of $A_1$ into $A_2$. Show the following:

1. For any term $t$ of $L$, $h(t^{I_1, \phi}) = t^{I_2, \phi_h}$.
2. For any $F$ not containing the equality symbol or quantifiers $\models_{A_1, \phi} F$ iff $\models_{A_2, \phi_h} F$.
3. If $h$ is a homomorphism of $A_1$ onto $A_2$, then for any formula $F$ not containing the equality symbol $\models_{A_1, \phi} F$ iff $\models_{A_2, \phi_h} F$. (Hint: The onto condition ensures that every object in one domain has a counterpart in the other domain.)
4. If $h$ is an isomorphism of $A_1$ onto $A_2$, then, for any formula $F$, $\models_{A_1, \phi} F$ iff $\models_{A_2, \phi_h} F$. (Hint: The isomorphism condition is needed for atomic formulas where equality is used.)