In this homework, the symbol \( \neg \) is used to represent negation instead of the symbol \( \sim \) used in class.

**Problem 1**

Let \( F[A_1, \ldots, A_n] \) denote a propositional formula whose propositional letters are contained in the set \( \{A_1, \ldots, A_n\} \) and let \( X_1, \ldots, X_n \) be \( n \) arbitrary propositional formulas. Then we denote by \( F[X_1, \ldots, X_n] \) the result of simultaneously replacing all occurrences of \( A_i \) by \( X_i \), for \( 1 \leq i \leq n \), in \( F[A_1, \ldots, A_n] \). Thus, if \( F[A, B] \) is \( ((A \land \top) \lor \neg B) \), then \( F[P \supset Q, P \lor R] \) is \( ((P \supset Q) \land \top) \lor \neg(P \lor R) \). Show that if \( F[A_1, \ldots, A_n] \) is a tautology, then so is \( F[X_1, \ldots, X_n] \). Hint: Show that for any interpretation that yields certain truth values for \( F[X_1, \ldots, X_n] \), there is an interpretation that yields the same truth values for \( F[A_1, \ldots, A_n] \). Use induction on the structure of \( F[A_1, \ldots, A_n] \) to show this property.

**Problem 2**

Using \( NKP \) and \( LKP \), show that the following are tautologies:

1. \( ((A \supset C) \supset ((B \supset C) \supset ((A \lor B) \supset C))) \)
2. \( ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)) \)
3. \( (((A \supset B) \supset A) \supset A) \)

Which of these is also provable in the intuitionistic versions of these calculi?

**Problem 3**

Prove the following:

1. The sequent \( A, \Gamma \rightarrow \Delta \) has an \( LKP \) proof if and only if the sequent \( \Gamma \rightarrow \neg A, \Delta \) has one.
2. The sequent \( \neg A, \Gamma \rightarrow \Delta \) has an \( LKP \) proof if and only if the sequent \( \Gamma \rightarrow A, \Delta \) has one.

**Problem 4**

Show that \( LKP \) is sound, \( i.e. \) that there is a proof tree for a sequent in \( LKP \) only if the sequent is valid. You will need to use induction on the height of a proof tree for this.

**Problem 5** (Completeness for \( LKP \))

Let \( LKP^* \) be the sequent calculus obtained from \( LKP \) by leaving out the \textit{Contraction} inference rule and replacing the \textit{\&\&-L} and \textit{\lor\lor-R} rules by the following:
\[
\begin{align*}
A, B, \Gamma \rightarrow & \Delta \\
A \land B, \Gamma \rightarrow & \Delta \land L^* \\
\Gamma \rightarrow & A, B, \Delta \\
\Gamma \rightarrow & A \lor B, \Delta \lor R^*
\end{align*}
\]

1. Show that corresponding to each proof tree in \(LKP^*\) there is one in \(LKP\) with the same label on the root, i.e., every sequent that has an \(LKP^*\) proof also has an \(LKP\) proof.

2. Show that if a sequent \(\Gamma \rightarrow \Delta\) is valid, then it has a proof in \(LKP^*\), i.e., that \(LKP^*\) is complete. (Hint: Use an induction on the number of logical symbols in \(\Gamma, \Delta\)).

3. Conclude from (1) and (2) that \(LKP\) is complete for propositional logic. In fact, conclude that a sequent is valid only if it has an \(LKP\) proof in which the only place where \textit{Contraction} is used is within the “derived” rules \(\land L^*\) and \(\lor R^*\).

**Problem 6**

Construct a proof system similar to \(LKP^*\) of Problem 4 but one in which the sequent symbol does not appear. As a first step in this direction, observe that instead of the sequent

\[
A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m
\]

you might use the multiset of formulas

\[
\neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m.
\]

Show that the new proof system that you have described is sound and complete (in appropriately defined senses) with respect to propositional logic. (Hint: Think of a rule corresponding to each rule in \(LKP^*\). Prove soundness and completeness like in Problems 3 and 4). You might be interested in comparing the proof system you develop in this problem with the semantic tableaux one that is discussed, for example, in Chapter 3 of Fitting’s book.

**Problem 7**

Suppose \(S\) is a set such that for any interpretation \(v\) there is at least one element of \(S\) which is true under \(v\). Show that there is a finite subset \(\{X_1, \ldots, X_n\}\) of \(S\) such that the disjunction \((X_1 \lor \ldots \lor X_n)\) is a tautology. (You will need to wait till we have talked about the compactness theorem in class for this one.)