Supplementary Material for "Uncertainty Models for TTC-Based Collision Avoidance"

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I. TTC Derivatives

In the TTC and UTTC models, τ is a solution to a quadratic equation, $a\tau^2 + 2b\tau + c = 0$, whose coefficients a, b, and c depend on $\mathbf{x}$ and $\mathbf{v}$. To compute the associated (U)TTC force, we require its derivative $\frac{\partial \tau}{\partial \mathbf{x}}$. Here, we compute this derivative explicitly in terms of the derivatives of the coefficients. For brevity, we will use the prime notation $(\cdot)'$ to denote the derivative with respect to $\mathbf{x}$, $\frac{\partial(\cdot)}{\partial \mathbf{x}}$.

We know that

$$\tau = \frac{-b \pm \sqrt{D}}{a}, \tag{1}$$

where $D = b^2 - ac$ is the discriminant of the quadratic. However, this form is inconvenient to differentiate directly. Instead, we apply implicit differentiation to the original quadratic equation, yielding

$$a'\tau^2 + 2a\tau\tau' + 2b'\tau + 2b\tau' + c' = 0. \tag{2}$$

Therefore,

$$\tau' = -\frac{a'\tau^2 + 2b'\tau + c'}{2(a\tau + b)} \tag{3}$$

$$= \mp\frac{1}{2\sqrt{D}}(a'\tau^2 + 2b'\tau + c'), \tag{4}$$

using the fact that $\tau = (-b \pm \sqrt{D})/a$.

For the standard TTC model, the coefficients of the quadratic equation are

$$a = \|\mathbf{v}\|^2, \qquad a' = 0, \tag{5}$$

$$b = \mathbf{x}\cdot\mathbf{v}, \qquad b' = \mathbf{v}, \tag{6}$$

$$c = \|\mathbf{x}\|^2 - r^2, \qquad c' = 2\mathbf{x}, \tag{7}$$

and so the derivative of τ is simply

$$\frac{\partial \tau}{\partial \mathbf{x}} = \mp\frac{1}{\sqrt{D}}\left(\mathbf{x} + \mathbf{v}\tau\right). \tag{8}$$

For the isotropic model,

$$a = \|\hat{\mathbf{v}}\|^2 - \epsilon^2, \qquad a' = 0, \tag{9}$$

$$b = \mathbf{x}\cdot\hat{\mathbf{v}} - r\epsilon, \qquad b' = \hat{\mathbf{v}}, \tag{10}$$

$$c = \|\mathbf{x}\|^2 - r^2, \qquad c' = 2\mathbf{x}, \tag{11}$$

so again we have

$$\frac{\partial \hat{\tau}}{\partial \mathbf{x}} = \mp\frac{1}{\sqrt{D}}(\mathbf{x} + \hat{\mathbf{v}}\tau). \tag{12}$$

II. Choice of Root

As τ is defined by a quadratic equation $a\tau^2 + 2b\tau + c = 0$, we generically have two choices of root. We will show that in all cases, we only need to choose the root given by the negative sign,

$$\tau = \frac{-b - \sqrt{D}}{a} = \frac{c}{-b + \sqrt{D}}. \tag{13}$$

At the initial condition $t = 0$, the agents must not be intersecting, so the constant term c is positive. The product of the roots is c/a, and so has the same sign as a. If a is positive, either both roots are positive and we should take the smaller one (using $-\sqrt{D}/a$), or both roots are negative and the choice is immaterial as either one will be end up being discarded. If a is negative, one root is positive and the other is negative, so we should take the positive one (using $-\sqrt{D}/a$).

As a consequence, the $\mp$ signs in the expressions for $\partial\tau/\partial\mathbf{x}$ in the previous section can be omitted, as they are always positive for the roots of interest.

III. Energy-Based Adversarial Model

In the adversarial model described in the main text, we choose $\tilde{\mathbf{v}} = \hat{\mathbf{v}} - \epsilon\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and prescribe the inter-agent interaction using the standard TTC force $\mathbf{f}_{\text{TTC}}(\mathbf{x}, \tilde{\mathbf{v}}, r)$. Unlike the standard TTC model and the isotropic model, this approach does not have the property of being derived directly from an interaction energy. If such a property is desired, one can define an energy $U_{\text{adv}} = f(\tau(\mathbf{x}, \tilde{\mathbf{v}}, r))$ and derive the force via its gradient. This yields an interaction model very similar, though not identical, to the adversarial model discussed in the text. For completeness, we give here the formula for this force, $\mathbf{f}_{\text{adv}} = -\partial U_{\text{adv}}/\partial\mathbf{x}$.

The time to collision $\hat{\tau}$ is now determined by

$$\|\mathbf{x} + \tilde{\mathbf{v}}\hat{\tau}\|^2 = r^2, \tag{14}$$

where $\tilde{\mathbf{v}} = \hat{\mathbf{v}} - \epsilon\frac{\mathbf{x}}{\|\mathbf{x}\|}$. The coefficients of this equation are

$$a = \|\tilde{\mathbf{v}}\|^2, \qquad a' = -\frac{2\epsilon}{\|\mathbf{x}\|}\left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}\right)\tilde{\mathbf{v}}, \tag{15}$$

$$b = \mathbf{x}\cdot\tilde{\mathbf{v}}, \qquad b' = \tilde{\mathbf{v}}, \tag{16}$$

$$c = \|\mathbf{x}\|^2 - R^2, \qquad c' = 2\mathbf{x}, \tag{17}$$

and so

$$\frac{\partial \tau}{\partial \mathbf{x}} = \frac{1}{\sqrt{D}}\left(\mathbf{x} + \tilde{\mathbf{v}}\tau - \frac{\epsilon}{\|\mathbf{x}\|}\left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}\right)\tilde{\mathbf{v}}\tau^2\right), \tag{18}$$

where $D = b^2 - ac = (\mathbf{x}\cdot\tilde{\mathbf{v}})^2 - \|\tilde{\mathbf{v}}\|^2(\|\mathbf{x}\|^2 - R^2)$.

IV. Proof of Lemma 3

We first restate the lemma:

Lemma 3. *If two agents collide at a time t^*, and all forces other than the TTC force $\mathbf{f}_{ij}$ are bounded, there exists an interval $[t_0, t^*]$ in which the TTC force between the agents performs an unbounded amount of negative work.*

Proof: As $s(t)$ approaches zero, the VO cone S_{TTC} becomes approximately a half-space, and the time to collision can be well approximated by $\tau(t) \approx -s(t)/s'(t)$. Because $s(t) = 0$ and $s'(t) < 0$ at t^*, there exists a time interval $[t_0, t^*]$ in which $\dot{s}$ is bounded away from zero. In this interval, $\tau(t)$ is finite and decreases to 0 at time t^*.

As the rate of work done by the TTC force is $\dot{W} = f'(\tau)$, the total work done over the interval $[t_0, t^*]$ is its integral,

$$W = \int_{t_0}^{t^*} f'(\tau(t))\,\mathrm{d}t.$$

Suppose we have the bounds $\tau'_* \le \tau'(t) < 0$. Then, using the fact that $\mathrm{d}f(\tau(t))/\mathrm{d}t = f'(\tau(t))\tau'(t)$, we can show that the integral is unbounded:

$$\begin{aligned} W &= \int_{t_0}^{t^*} \frac{\mathrm{d}f(\tau(t))/\mathrm{d}t}{\tau'(t)}\,\mathrm{d}t \\ &\le \frac{1}{\tau'_*} f(\tau(t))\Big|_{t=t_0}^{t=t^*} \\ &= -\infty \end{aligned}$$

because $f(\tau(t^*)) = f(0) = \infty$.

If the nonpositivity assumption $\tau'(t) < 0$ is violated for some values of t, we can simply ignore them in the integration domain and apply the same argument to show that the reduced integral is still unbounded. It is however essential that a lower bound $\tau'_* \le \tau'(t)$ holds. Differentiating $\tau(t) = -s(t)/s'(t)$, we obtain that

$$\tau'(t) = -1 + \frac{s(t)s''(t)}{s'(t)^2}.$$

Given that $s'(t)$ is bounded away from zero in the interval of interest, $\tau'(t)$ can be unbounded below only if the same is true of $s''(t)$; that is, there exists an unbounded force pushing the agents *together*. This contradicts our assumption that the other forces in the system are bounded, because the TTC force is always repulsive. ■