Correlated Equilibria in Graphical Games

by
Sham Kakade
Michael Kearns
John Langford
Luis Ortiz

presented by
Steven Damer
Outline

- Graphical Games
- Markov Nets
- Finding Correlated Equilibria
Graphical Games

• A *Graphical Game* is a pair \((G, M)\), where \(G\) is an undirected graph over the vertices \(\{1 \ldots n\}\), and \(M\) is a set of \(n\) local game matrices. For any joint action \(\vec{a}\), the local game matrix \(M_i \in M\) specifies the payoff \(M_i(\vec{a}^i)\) for player \(i\), which depends only on the actions taken by the players in \(N(i)\).

• Graphical games are a subset of all \(n\)-player games which take advantage of restrictions between player interactions to achieve a smaller representation.

• A general \(n\)-player game requires space exponential in \(n\), whereas a graphical game requires space exponential in \(d\) where \(d\) is the size of the largest neighborhood in \(G\).
Equivalences

- Two distributions $P$ and $Q$ over joint actions $\bar{a}$ are *expected payoff equivalent*, denoted by $P \equiv_{EP} Q$, if $P$ and $Q$ yield the same expected payoff vector: for each $i$, $E_{\bar{a} \sim P}[M_i(\bar{a}^i)] = E_{\bar{a} \sim Q}[M_i(\bar{a}^i)]$.

- Two distributions are expected payoff equivalent if they yield the same payoff to all players.

- For a graph $G$, two distributions $P$ and $Q$ over joint actions $\bar{a}$ are *local neighborhood equivalent* with respect to $G$, denoted by $P \equiv_{LN} Q$, if for all players $i$, and for all settings $\bar{a}^i$ of $N(i)$, $P(\bar{a}^i) = Q(\bar{a}^i)$.

- Two distributions are local neighborhood equivalent if they preserve correlations between values at the neighborhood level.
Equivalence Theorems

- For all graphs $G$, for all joint distributions $P$ and $Q$ on actions, and for all graphical games with graph $G$, if $P \equiv_{LN} Q$ then $P \equiv_{EP} Q$. Furthermore, there exists payoff matrices $M$ such that for the graphical game $(G, M)$, if $P \not\equiv_{LN} Q$ then $P \not\equiv_{EP} Q$.

- Local neighborhood equivalence implies expected payoff equivalence, and if local neighborhood equivalence doesn’t hold, then there exist games for which expected payoff equivalence doesn’t hold either.

- For any graphical game $(G, M)$ if $P \in CE(G, M)$ and $P \equiv_{LN} Q$ then $Q \in CE(G, M)$.

- If two distributions are local neighborhood equivalent, then either they both are correlated equilibria or neither are.
Local Markov Networks

• A local Markov network is a pair $M \equiv (G, \Psi)$ where
  
  1. $G$ is an undirected graph on vertices $\{1, \ldots, n\}$;
  2. $\Psi$ is a set of potential functions, one for each local neighborhood $N(i)$, mapping binary assignments of values of $N(i)$ to the range $[0, \infty)$:
      $$\Psi \equiv \{\psi_i : \{\Vec{a}^i\} \rightarrow [0, \infty)\}$$

Here $\{\Vec{a}^i\}$ is simply the set of all $2^{|N(i)|}$ settings to $N(i)$.

• A local Markov network $M$ defines a probability distribution $P_M$ as:

$$P_M(\Vec{a}) \equiv \frac{1}{Z} \left( \prod_{i=1}^{n} \psi_i(\Vec{a}^i) \right)$$

where $Z = \sum_{\Vec{a}} \prod_{i=1}^{n} \psi_i(\Vec{a}^i) > 0$ is the normalization factor
Markov networks and correlated equilibria

- For all graphs $G$, and for all joint distributions $P$ over join actions, there exists a distribution $Q$ that is representable as a local Markov network with graph $G$ such that $Q \equiv_{LN} P$ with respect to $G$.

- Every joint distribution is representable by a local neighborhood equivalent local Markov network.

- For all graphical games $(G, M)$, and for all distributions $P \in CE(G, M)$, there exists a distribution $Q$ such that:
  1. $Q \in CE(G, M)$
  2. $Q \equiv_{EP} P$
  3. $Q$ can be represented as a local Markov network with graph $G$.

- Every correlated equilibrium is expected payoff equivalent to another correlated equilibrium which can be represented as a local Markov network.
Finding Correlated Equilibria in Graphical Games

- Linear programming can find correlated equilibria for arbitrary games in time polynomial to the size of the representation. Unfortunately, the representation is exponential in the number of players.

- Graphical games can be used to simplify the linear programming problem by reducing the size of the representation.
Constraints

- For every player $i$ and every assignment $\vec{a}^i$, there is a variable $P_i(\vec{a}^i)$ with the following constraints:
  
  - CE Constraints: for all players $i$ and actions $a, a'$

  $$\sum_{\vec{a}^i:a_i^i=a} P_i(\vec{a}^i)M_i(\vec{a}^i) \geq \sum_{\vec{a}^i:a_i^i=a} P_i(\vec{a}^i)M_i([\vec{a}^i[i:a']]$$

  - Neighborhood Marginal Constraints: for all players $i$

  $$\forall \vec{a}^i : P_i(\vec{a}^i) \geq 0 \land \sum_{\vec{a}^i} P_i(\vec{a}^i) = 1$$

  - Intersection Consistency Constraints: for all players $i$ and $j$, and for any assignment $\vec{y}^{i,j}$ to the intersection neighborhood $N(i) \cap N(j)$

  $$P_i(\vec{a}^{i,j}) = P_j(\vec{a}^{i,j})$$

- This system involves only $O(n^2d)$ variables and inequalities, which is a huge improvement over $O(2^n)$
Local Correlated Equilibria

- For all graphical games \((G, M)\) in which \(G = (V, E)\) is a tree, and for all assignments \(\{P_i(\bar{a}^i)\}\) satisfying the consistency equations, there is a unique joint distribution \(Q\) defined by
  \[
  Q(\bar{a}^i) \equiv \frac{\Pi_{i \in V} P_i(\bar{a}^i)}{\Pi_{(i,j) \in E; i < j} P_i(\bar{a}^{ij})}
  \]
  such that \(Q \in CE(G, M)\) and \(Q\) is representable as a local Markov network with graph \(G\). Furthermore, the marginals of \(Q\) will be consistent with the assignment \(\forall i, \bar{a}^i, Q_i(\bar{a}^i) = P_i(\bar{a}^i)\).

- For all graphical games which are trees there is a correlated equilibrium which is representable as a local Markov network.
Completeness Theorem

- For all graphical games \((G, M)\) such that \(G\) is a tree, we have \(CE_{local}(G, M) \subseteq CE(G, M)\); and if \(P \in CE(G, M)\) then there exists a \(Q \in CE_{local}(G, M)\) such that \(P \equiv_{EP} Q\).

- For every correlated equilibrium of a graphical game based in a tree there is an expected payoff equivalent correlated equilibrium which can be represented with a local Markov network.
Efficient Tree Algorithm

- For all tree graphical games \((G, M)\) and all linear objective functions \(F(\{P_i(\vec{a}^i)\})\), linear programming computes a CE in time polynomial in the size of the graphical game. The CE computed can be varied by varying the objective function \(F\).

- Sample objective function:

\[
\max_{\text{subject to constraint on } P_i(\vec{a}^i)} \sum_i \sum_{\vec{a}^i} P_i(\vec{a}^i) M_i(\vec{a}^i)
\]

- You can find a correlated equilibrium for a graphical game on a tree in a reasonable amount of time.
Thoughts

- Local Markov networks aren’t suitable to games on a grid, but Markov networks might be.

- Graphical games provide a way to determine which correlations in a correlated equilibrium are essential.

- More computationally expensive than learning the equilibrium, but it converges instantly (but this means you might not select the same equilibrium as your opponents).