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GRADUATE SCHOOL

The Suspension Calculus and its Relationship to Other Explicit Treatments of Substitution in Lambda Calculi

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Abstract

The intrinsic treatment of binding in the lambda calculus makes it an ideal data structure for representing syntactic objects with binding such as formulas, proofs, types, and programs. Supporting such a data structure in an implementation is made difficult by the complexity of the substitution operation relative to lambda terms. To remedy this, researchers have suggested representing the meta level substitution operation explicitly in a refined treatment of the lambda calculus. The benefit of an explicit representation is that it allows for a fine-grained control over the substitution process, leading also to the ability to intermingle substitution with other operations on lambda terms. This insight has lead to the development of various explicit substitution calculi and to their exploitation in new algorithms for operations such as higher-order unification. Considerable care is needed, however, in designing explicit substitution calculi since within them the usually implicit operations related to substitution can interact in unexpected ways with notions of reduction from standard treatments of the lambda calculus.

This thesis describes a particular realization of explicit substitutions known as the suspension calculus and shows that it has many properties that are useful in a computational setting. One significant property is the ability to combine substitutions. An earlier version of the suspension calculus has such an ability, but the complexity of the machinery realizing it in a complete form has deterred its direct use in implementations. To overcome this drawback a derived version of the calculus had been developed and used in practice. Unfortunately, the derived calculus sacrifices generality and loses a property that is important for new approaches to unification. This thesis redresses this situation by presenting a modified form of the substitution combination mechanism that retains the generality and the computational properties of the original calculus while being simple enough to use directly in implementations. These modifications also rationalize the structure of the calculus, making it possible to easily superimpose additional logical structure over it. We illustrate this capability by showing how typing in the lambda calculus can be treated in the resulting framework and by presenting a natural translation into the $\lambda\sigma$-calculus, another well-known treatment of explicit substitutions.

Another contribution of this thesis is a survey of the realm of explicit substitution calculi. In particular, we describe the computational properties that are desired in this setting and then characterize various calculi based on how well they capture these. We utilize the simplified suspension calculus in this process. In particular, we describe translations between the other popular calculi and the suspension calculus towards understanding and contrasting their relative capabilities. Finally, we discuss an elusive property of explicit substitution calculi known as preservation of strong normalization and discuss why there is hope that the suspension calculus possesses this property.

Contents

Chapter 1

Introduction

Binding and scoping of variable names is a fundamental concept in mathematics and computer science. Consider the mathematical statement that the natural numbers have no largest element,

$$\forall x \in \mathbb{N}.\ \exists y \in \mathbb{N}.\ y > x$$

In this statement the occurrences of x and y on the right are bound by the quantifiers on the left, and the scoping of these variables is important. Picking y as the value for x and substituting blindly would yield,

$$\exists y \in \mathbb{N}.\ y > y$$

But this statement is no longer true because the substitution was not done correctly. The problem is that y is not yet in scope when the quantifier for x appears, thus any substitution for x cannot contain the variable y. This situation is replayed in the following computer science context:

```
int x = 3;
int y = x + 2;
```

Here the variables `x` and `y` are bound by declaration of `int x` and `int y`, and here again the scoping is important: we can use `x` in defining `y` but not vice-versa. The point of these examples is that if we are going to represent and manipulate objects from mathematics and computer science, then we should use a language which has an understanding of binding and scoping.

1.1 The Lambda Calculus and Its Treatment of Binding

The lambda calculus is a notation for functions which correctly captures notions of binding and scoping [Chu40]. For example, the function which maps x to $x + x$ is encoded as

$(\lambda x.x + x)$, and applying this function to the argument 3 is encoded as $((\lambda x.x + x)\ 3)$. We expect that this expression is equal to $3+3$, and indeed the lambda calculus formalizes this notion of equality so that the equation $((\lambda x.x+x)\ 3) = 3+3$ holds. Binding in the lambda calculus is done such that variable occurrences are bound by the closest enclosing binder which matches the variable name. For instance, in $(\lambda x.\lambda x.x + x)$ the two occurrences of x on the right are bound by second lambda. Substitution in the lambda calculus is capture avoiding in the sense that a free variable, one which is not bound by a lambda, cannot become bound by the process of substitution. Thus it is *not* true that $((\lambda x.\lambda y.x)\ y) = (\lambda y.y)$. Instead, the bound variable y is first renamed before the free variable y is substituted which yields the true equality $((\lambda x.\lambda y.x)\ y) = (\lambda z.y)$. Finally note that we could have picked a name other than z here so long as it did not capture the free variable y, because renaming of bound variables is an intrinsic property of equality in the lambda calculus.

Given this notion of equality, we can think of assigning directionality to it so that $((\lambda x.x + x)\ 3) \rightarrow 3 + 3$. This directionality gives rise to a notion of computation through function evaluation, which is common to most programming languages. Based on this idea, Landin argues that we can understand any particular programming language by translating it into the lambda calculus [Lan66]. Such a translation actually turns out to be a useful means for providing denotational semantics for programming languages based on the model theory for the lambda calculus developed by Plotkin, Scott, and Strachey [Sto81]. Another benefit of thinking of the lambda calculus as a common substrate for programming languages is that the essence of issues such as typing and evaluation strategies can be studied without the distraction of auxiliary features and nuances of any particular programming language.

A more recent development related to the lambda calculus is its use in representing syntactic objects whose structure incorporates some form of binding. The motivation behind this is that common notions of binding in these settings can be handled using the lambda calculus as a data structure. For example, the content of quantifiers such as "for all" and "there exists" in mathematical logic can be separated into two parts: one part which is the binding of a variable and another part which is predicative in nature. The predicative part can be represented by an appropriate constant such as $forall$ or $exists$ and the binding part can be represented by a lambda abstraction. Using these ideas, the formula $\forall x \exists y (y > x)$ considered earlier in this chapter can be represented by the lambda calculus expression

$$forall\ (\lambda x.\ exists\ (\lambda y.\ gt\ y\ x))$$

where gt is a constant representing the "greater than" relation. Using this representation, logical operations related to binding can be performed using the lambda calculus. For example, if we want to substitute y for x in the equation

$$\forall x \in \mathbb{N}.\ \exists y \in \mathbb{N}.\ y > x$$

We do this by applying the argument of our $forall$ constant to the variable y and then performing a reduction in the lambda calculus

$$(\lambda x.\ exists\ (\lambda y.\ gt\ y\ x))\ y \rightarrow exists\ (\lambda z.\ gt\ z\ y)$$

This result corresponds to the logical formula

$$\exists z \in \mathbb{N}.\ z > y$$

Similarity we may wish to identify our original statement with the statement

$$\forall a \in \mathbb{N}.\ \exists b \in \mathbb{N}.\ b > a$$

and this is provided for free using lambda calculus.

Similar notions of binding and substitution occur when representing many different kinds of objects such as formulas, proofs, types, and programs. The idea is that we can use a single meta language, the lambda calculus, to talk about all of these different objects. Then the correctness concerns of binding and substitution only need to be handled once, in this meta language, and then the benefit is shared in all other contexts.

1.2 The Explicit Treatment of Substitution

At the heart of the lambda calculus is a notion of substitution which respects the binding structure of functions in the language. The traditional presentation of the lambda calculus takes this substitution as a meta operation, which is impractical from an implementation perspective. Because of this, various methods have been developed for dealing with substitution in actual implementations. In the computational setting, what is often done is that an environment is kept which contains substitutions for variables. This approach has been successful in practice, but it restricts the possible evaluation strategies, since it expects that every variable encountered has a substitution available in the environment. For example, this expectation will not hold if we do evaluations underneath lambda abstractions since the abstracted variables have no substitutions.

In the representational setting, substitution is also a problem. Here we often perform unification modulo the rules of the lambda calculus, and the handling of substitution has a significant impact on the efficiency of this operation. Consider, for example, the unification $(\lambda x.c\ t_1)\ t_2 \stackrel{?}{=} d\ t_3$ for distinct constants c and d and some terms t_1, t_2, and t_3. A naive approach requires substituting the term t_2 for x throughout the the term t_1, which can be arbitrarily large. Instead, a more sophisticated technique is to treat substitution explicitly and include it as part of the language. Then we can reduce this unification problem to $c\ t_1\langle t_2/x\rangle \stackrel{?}{=} d\ t_3$ where $\langle t_2/x\rangle$ encodes a substitution that is part of the language and not a meta operation. At this point we can answer "no" since c and d are distinct constants, and thus we avoid traversing the term t_1.

Explicit treatment of substitution introduces its own difficulties. One particular difficulty stems from our interest in identifying expressions that differ only in the names of bound variables. To accommodate this, we use a nameless notation for the lambda calculus invented by de Bruijn [dB72]. In this notation, we replace each variable occurrence with an integer which counts the number of lambda abstractions between the occurrence and its binder. For example, the lambda term $(\lambda x.\lambda y.x\ x\ y)$ is encoded as $(\lambda\,\lambda\,2\ 2\ 1)$. This provides a unique encoding of terms differing only in bound variable renaming, but now substitution must take into account renumberings when substituting underneath a lambda abstraction. For example, $(\lambda\,\lambda\,((\lambda\,\lambda\,2)\ 2)) = (\lambda\,\lambda\,\lambda\,3)$. Thus any system for explicit substitutions must keep track of the renumbering to be done.

The explicit treatment of substitutions also gives rise to new benefits in the lambda calculus. One is that we may think of merging substitutions to decrease the amount of work done in traversing a term. For example, reducing $((\lambda x.\lambda y.t_1)\ t_2\ t_3)$ traditionally requires two traversals over the term t_1, but if the substitutions generated by reducing the above term are explicit then we can think of merging them into a single substitution before their effects are propagated onto t_1. Another benefit is that existing procedures such as unification can be improved upon by mixing their operations with the operations of explicit substitutions [DHK95].

These motivations for the explicit treatment of substitutions have been recognized by researchers, resulting in a wide variety of explicit substitution calculi [Fie90, NW90, ACCL91, KR95, BBLRD96, KR97, DG01b]. This thesis focuses on the suspension calculus [NW90].

1.3 Contributions of the Thesis

In this thesis we use the suspension calculus as a viewpoint into the realm of explicit substitution calculi. The particular contributions of this thesis are

A simplification and rationalization of the suspension calculus
The suspension calculus includes not only an explicit representation of substitutions, but also a mechanism for combining such substitutions. This combining is realized through a seemingly complex set of rules, and this apparent complexity has lead to the development of derived calculi [Nad99]. These derived calculi work well for reduction, but they lack an essential property and therefore they cannot be used to perform unification in a setting with a special interpretation of instantiable variables. This thesis offers another possibility by simplifying these combination rules so that the full calculus is practical to use in implementations, while retaining all of the essential properties of the original suspension calculus. These changes have the added benefit of rationalizing the calculus so that a typed version of the calculus is possible, and translations to other calculi are now feasible.

A comparison of explicit substitution calculi
A wide variety of explicit substitution calculi have been proposed, but no systematic attempt has been made to compare these calculi and analyze their essential features. In this thesis we outline desirable properties for explicit substitution calculi and use these to organize a survey of the more popular calculi. We also explore the relationship between these calculi and the suspension calculus by defining explicit translations for expressions. These translations serve as a means of understanding the notation and rules of each calculus using the framework of the suspension calculus.

1.4 Outline of the Thesis

The rest of this thesis is organized as follows. In the next chapter we review the lambda calculus and introduce the terminology associated with it that we will use in later chapters. We then describe the suspension calculus in Chapter 3 and prove its key properties. In Chapter 4 we survey other explicit substitution calculi and compare them with the suspension calculus. Then in Chapter 5 we discuss the property of preservation of strong normalization, which is an important issue for explicit substitution calculi. Finally, in Chapter 6 we review the contributions of this thesis and discuss possible avenues for future research.

Chapter 2

The Lambda Calculus

The lambda calculus is a language of functions—a simple and concise syntax for describing a powerful and expressive language. As discussed in the introduction, this calculus has two related uses at a practical level: (1) to represent syntactic objects that have a functional structure in a way that captures their functionality and (2) by interpreting representations of functions essentially as rules for calculations, it also supports the ability to encode computations in a fundamental way. In short, we call these two uses "representation" and "computation," respectively. The goal of this chapter is to define the calculus and present notation and properties that support these two different uses.

In the first section, we define the lambda calculus and various notions of equality on lambda terms. Following that, we look at the de Bruijn notation for the lambda calculus which precisely captures the most fundamental of these equality notions. In the third section, we look at properties of the lambda calculus which are important to its consistent and meaningful use in the roles of representation and computation. Finally, we look at adding instantiable variables to the calculus to support higher-order unification in the representational setting. The results presented in this chapter are well established in the literature and proven, for instance, in [Bar81].

2.1 The Syntax and Meaning of Lambda Terms

Is $x + x$ a function? The answer depends on the context. In an equation such as $x + x = 4$, we think of $x + x$ as a value, not a function. On the other hand, in the statement, "$x + x$ is primitive recursive," we are thinking of the function which maps x to $x + x$. Church resolved this ambiguity by introducing an explicit notation for functions, the lambda calculus [Chu40]. In the lambda calculus we denote a function mapping x to $x + x$ by $(\lambda x.x + x)$. The lambda in this expression creates an abstraction over x so that x is a bound variable within the subexpression $x + x$. Juxtaposition denotes application, and so $((\lambda x.x + x)\ 5)$ is the application of our function to the argument 5. Later in this section we will introduce notions of equality that capture our intuition about function equality and evaluation. Surprisingly, abstraction and application along with these notions of equality can model all computable functions.

With the syntax above, we can now use functions in a first-class way. That is, we can use functions not only to manipulate input and produce output, but also as the input and output of other functions. Consider the function $(\lambda x.(\lambda y.x + y))$. This is a function which when applied to some input, produces another function as output. This allows us to encode functions of more than one argument by nesting functions of exactly one argument. As another example, consider $(\lambda f.(\lambda x.f\ x))$. This function takes a function f and an argument x and returns the application of f to x. In this instance, we are thinking of a function as input to another function. In this way, functions in the lambda calculus are first-class and higher-order—an expressive notion captured by simple syntax.

2.1.1 Syntax

Formally, the terms in the lambda calculus are defined by the following.

Definition 2.1.1 (Lambda terms)**.** *Terms in the lambda calculus are defined by*

$$t ::= c \mid x \mid (t\ t) \mid (\lambda x.t)$$

where c ranges over some enumerable set of constants and x over some enumerable set of variables.

We call $(t\ t)$ an application and $(\lambda x.t)$ an abstraction. To differentiate between variables and constants, we denote constants by letters like a, b, c or by appropriate symbols, and we denote variables by letters like x, y, z. To reduce the number of parenthesis we need to write, we follow the convention that application is left associative and the scope of lambda extends as far to the right as possible. For example, $(\lambda x.((x\ y)\ z))$ is written as $(\lambda x.x\ y\ z)$. We will sometimes include optional parenthesis when they aid readability.

The syntax in Definition 2.1.1 is a solid starting point for the lambda calculus, but there are many notions we need to build on top of this. First, the definition of syntax yields an obvious definition of subterm which we will assume. Second, the syntax of the lambda calculus contains a notion of binding which we must define explicitly. This is crucial because the binding structure of a term is its most essential part. All variable occurrences within a term are either free or bound. The basic rule of binding is that a variable occurrence of x is free if and only if it does not occur within the scope of a λx, and all free occurrences of the variable x in the term t are bound by the lambda in $\lambda x.t$.

When picking names for bound variables, we must be careful that they do not conflict with the free variables in a term. Thus the following definition will be very useful.

Definition 2.1.2 (Free variables)**.** *The set of free variables of a lambda term t, denoted $fv(t)$ is defined recursively as follows,*

$$\begin{aligned} fv(x) &= \{x\} \\ fv(\lambda x.t) &= fv(t)\backslash\{x\} \\ fv(t_1\ t_2) &= fv(t_1) \cup fv(t_2) \end{aligned}$$

If $fv(t) = \emptyset$ then t is called a closed term.

Now that we have a clear notion of the binding structure of terms, we can relate terms based on this structure. In the next section, we look at notions of equality based on the binding structure of terms.

2.1.2 Equality and Equivalence

The introduction of the lambda calculus as a language of functions gives us some preconceived notions of how lambda terms are related. One is that we expect the specific variable names used for bound variables to be irrelevant. Concretely, we consider the terms $(\lambda x.x + x)$ and $(\lambda y.y + y)$ equal, because they represent the same function.

We also expect a notion of equality under function evaluation. So the terms $((\lambda x.x{+}x)\ 5)$ and $5 + 5$ should be equal in this sense. Informally, we want to say that the term $(\lambda x.t_1)\ t_2$ is equal to the term $t_1[t_2/x]$ where $[t_2/x]$ is an operator which replaces all free occurrences of x in t_1 with t_2. This substitution operator must respect the binding structure of both t_1 and t_2. Specifically, we respect the structure of t_1 by not substituting t_2 in for any bound occurrences x, and we respect the structure of t_2 by not allowing any free variables in t_2 to become bound in t_1. These restrictions are what gives rise to the various branching conditions in the following definition.

Definition 2.1.3 (Substitution)**.** *The substitution operation $[s/x]$ which replaces the vari-*

able x with the term s is defined recursively as

$$
\begin{aligned}
c[s/x] &= c \\
y[s/x] &= \begin{cases} s & \text{if } x = y \\ y & \text{otherwise} \end{cases} \\
(t_1\ t_2)[s/x] &= (t_1[s/x])\ (t_2[s/x]) \\
(\lambda y.t)[s/x] &= \begin{cases} \lambda y.t & \text{if } x = y \\ \lambda y'.(t[y'/y][s/x]) & \text{if } y \in fv(s), \text{ where } y' \notin fv(y) \text{ and } y' \neq x \\ \lambda y.(t[s/x]) & \text{otherwise} \end{cases}
\end{aligned}
$$

The middle case for $(\lambda y.t)[s/x]$ takes advantage of our notion of equality for terms that differ only in the names of bound variables. In this case, we do not want a free occurrence of y in s to be captured by the binding lambda, so we rename the bound variable before performing the substitution.

With a definition of substitution in place, we can go back and formally define what we mean by terms that differ only in the names of bound variables.

Definition 2.1.4 (α-equivalence)**.** *A term s results from a term r by α-conversion if s can be obtained from r by replacing some subterm of the form $\lambda x.t$ by one of the form $\lambda y.(t[y/x])$ where y is a variable that is not free in t. Two terms s and r are said to be α-equivalent, written as $s =_\alpha r$, if one can be obtained from the other by a (possibly empty) sequence of α-conversions.*

We can also formalize a notion of equivalence under evaluation that we suggested before.

Definition 2.1.5 (β-equivalence)**.** *A term s results from a term r by a β-contraction, denoted $r \rhd_\beta s$, if s can be obtained by replacing a subterm of r of the form $(\lambda x.t_1)\ t_2$, referred to as a β-redex, by $t_1[t_2/x]$. We say also that s results from r by a β-reduction, denoted $r \rhd_\beta^* s$, if it can be obtained from r by a (possibly empty) sequence of α-conversions and β-contractions. The term r is said to result from s by a β-expansion if s results from r by a β-contraction. Finally, r and s are said to be β-equivalent of one results from the other by a sequence of α-conversions, β-contractions, and β-expansions, and we denote this by $r =_\beta s$.*

Note that β-contraction invokes substitution which may cause the renaming of some bound variables. For this reason we include α-conversion in our definition of β-reduction and β-equivalence.

A final notion of equality that we might expect from functions is that of extensional equivalence. That is, given $r\ x =_\beta s\ x$ we might expect a notion of equality that says r and s are equal. First note that β-equivalence is not powerful enough for this. For instance, the terms $(\lambda x.f\ x)$ and f are related in this way, but they are not β-equivalent. It turns out that we can get this extensionality property through the following equivalence notion.

Definition 2.1.6 (η-equivalence)**.** *A term s results from a term r by a η-contraction if s can be obtained by replacing a subterm of r of the form $(\lambda x.f\ x)$, referred to as a η-redex, by f, where x is not free in f. The term r is said to result from s by a η-expansion if s results from r by a η-contraction. Finally, r and s are said to be η-equivalent of one results from the other by a sequence of α-conversions, β-conversions, η-contractions, and η-expansions, and we denote this by $r =_\eta s$.*

Throughout this section we have referred to our equality notions as equivalence relations, and the following theorem justifies this.

Theorem 2.1.1. *$=_\alpha$, $=_\beta$, and $=_\eta$ are equivalence relations.*

2.2 De Bruijn Notation

Substitution in the lambda calculus is complicated by the possibility of variable names conflicting. De Bruijn notation is a nameless notation for the lambda calculus which abstracts away many of these issues [dB72]. The purpose of variable names in the lambda calculus is to associate variable occurrences with their binder. The de Bruijn notation makes this association by counting the number of lambdas that occur between a variable occurrence and its binder in the abstract syntax tree. In this way, names are removed both from variable occurrences and from binders. For example, the term $(\lambda x.(\lambda y.y)\ (\lambda z.x))$ is encoded as $(\lambda\,(\lambda\,\#1)\ (\lambda\,\#2))$. This term has the same content as the original, but we have abstracted away the specific variable names.

2.2.1 Terms in the De Bruijn Notation

Formally, the terms in the lambda calculus are defined by the following.

Definition 2.2.1 (De Bruijn terms)**.** *Terms in the de Bruijn notation are defined by*

$$t ::= c \mid \#i \mid (t\ t) \mid (\lambda\, t)$$

where c ranges over an enumerable set of constants and i, called an index or variable reference, ranges over the natural numbers.

As with the definition of lambda terms, we call $(t\ t)$ an application and $(\lambda\, t)$ an abstraction. We also drop parenthesis by assuming application is left associative and the scope of a lambda extends as far right as possible. In addition, we assume the obvious definition of subterm.

One issue which we have not yet clarified is how to deal with free variables. Because free variables have no binders associated with them, it is not obvious how to assign them an index. To handle this we will assume a fixed listing of the free variables of a term, which we think of as a list of top level binders for the free variables. Thus the term $(\lambda x.y\ x)$ where y is free will be encoded as $(\lambda\, \#2\ \#1)$. In this term, the $\#2$ refers to the first free variable.

2.2.2 Conversion and Equality in the De Bruijn Notation

A pleasant property of the de Bruijn notation is that we get α-equivalence for free, *i.e.*, any two terms which are α-equivalent in the lambda calculus have the same representation in the de Bruijn notation. Thus the complicated α-equivalence check in the lambda calculus has been replaced by a simple syntactic equality check in the de Bruijn notation.

We must also reconsider β-contraction in this notation. Given the de Bruijn β-redex $((\lambda\, M)\ N)$ we want to think about substituting N for the first free variable in M. But in performing this contraction, we have also eliminated a lambda which was previously over the term M. Thus all the free variables in M will have to have their index decremented by one. Also, we may have to substitute N beneath some lambdas which will require us to renumber all the free variables in N. This leads us to consider a generalized notion of substitution which allows us to substitute for every free variable in a term.

Definition 2.2.2 (De Bruijn substitution)**.** *Let t be a de Bruijn term and let $s_1, s_2, s_3 \ldots$ be an infinite sequence of de Bruijn terms. The result of simultaneously substituting s_i for the ith free variable in t is denoted by $S(t; s_1, s_2, s_3, \ldots)$ and is define by,*

1. *$S(c; s_1, s_2, s_3, \ldots) = c$, for any constant c,*
2. *$S(\#i; s_1, s_2, s_3, \ldots) = s_i$, for any index $\#i$,*
3. *$S((t_1\ t_2); s_1, s_2, s_3, \ldots) = S(t_1; s_1, s_2, s_3, \ldots)\ S(t_2; s_1, s_2, s_3, \ldots)$, and*
4. *$S((\lambda\, t); s_1, s_2, s_3, \ldots) = \lambda\, S(t; \#1, s'_1, s'_2, s'_3, \ldots)$ where $s'_i = S(s_i; \#2, \#3, \#4, \ldots)$.*

The interesting case in this definition is when we descend underneath an abstraction. Within this abstraction, the index #1 should be left untouched, the index #2 should refer to what we were substituting for the first free variable, the index #3 should refer to what we were substituting for the second free variable, etc. Further, since the arguments being substituted in are going to be placed under this additional lambda, we need to increment the indices of all free variables in those arguments. Given this extended definition of substitution, we can define β-conversion for the de Bruijn notation.

Definition 2.2.3 (De Bruijn β-conversion)**.** *A term s results from a term r by a β-contraction, denoted $r \rhd_\beta s$, if s can be obtained by replacing a subterm of r of the form $((\lambda\, t_1)\ t_2)$, referred to as a β-redex, by $S(t_1; t_2, \#1, \#2, \ldots)$. We say also that s results from r by a β-reduction, denoted $r \rhd_\beta^* s$, if it can be obtained from r by a (possibly empty) sequence of β-contractions. The term r is said to result from s by a β-expansion if s results from r by a β-contraction. Finally, r and s are said to be β-equivalent of one results from the other by a sequence of β-contractions and β-expansions, and we denote this by $r =_\beta s$.*

While convenient from the implementation standpoint, the de Bruijn notation is not particularly readable for humans. For example, the term $(\lambda x.(\lambda y.(\lambda z.x)\ x)\ x)$ is encoded as $(\lambda\,(\lambda\,(\lambda\,\#3)\ \#2)\ \#1)$. A naive glace at this term may suggest that three different variables are referenced, even though the same variable is referenced each time. Similarly, the term $(\lambda x.(\lambda y.(\lambda z.z)\ y)\ x)$ is encoded as $(\lambda\,(\lambda\,(\lambda\,\#1)\ \#1)\ \#1)$. Again, a naive glace may suggest that the same variable is referenced three times when in fact each reference is to a different variable. For the rest of this thesis we shall use the de Bruijn notation, but because of the reasons stated above we will often present examples using the named lambda calculus.

2.3 Properties of the Lambda Calculus

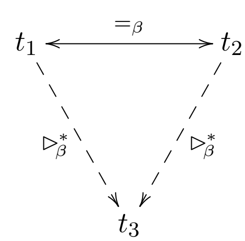
The β-equivalence rule performs the "heavy-lifting" of the lambda calculus. In the computational setting we use β-equivalence to determine the value of a computation, and in the representational setting we use β-equivalence to determine if two terms represent the same thing. In this section we look at properties of the lambda calculus which make it suitable for use in both of these settings.

2.3.1 β-equivalence and Confluence

Determining β-equivalence is seemingly difficult because we can use both β-contraction and β-expansion. Using β-expansion is impractical since we can apply it anywhere in a term, thus we will try to restrict ourselves to β-contraction which limits us to considering only the β-redexes of a term. In making this restriction, we may fear that we lose completeness, *i.e.*, that two terms are β-equivalent, but there is no common term to which they β-reduce. The next theorem addresses this fear and assures us that such a situation cannot occur.

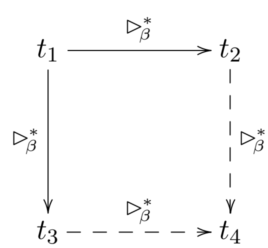
Before we can state the theorem, we need to introduce a diagram notation which is common in rewriting systems such as the lambda calculus. In such systems, we often have statements of the form "Let P and Q holds, then R and S are true" where P, Q, R, and S denote some relationships between terms. We represent this in a diagram by drawing solid arrows for the given properties, P and Q, and using dashed arrows for the resulting properties, R and S.

Theorem 2.3.1 (Church-Rosser property)**.** *Let t_1 and t_2 be lambda terms such that $t_1 =_\beta t_2$. Then there exists a term t_3 such that $t_1 \rhd_\beta^* t_3$ and $t_2 \rhd_\beta^* t_3$,* i.e.*, the follow diagram holds.*



This theorem tells us that there is no gap in completeness if we restrict ourselves to β-reduction when determining β-equivalence. This result is equivalent to the following property,

Theorem 2.3.2 (Confluence)**.** *Let t_1, t_2, t_3 be lambda terms such that $t_1 \rhd_\beta^* t_2$ and $t_1 \rhd_\beta^* t_3$. Then there exists a term t_4 such that $t_2 \rhd_\beta^* t_4$ and $t_3 \rhd_\beta^* t_4$,* i.e.*, the following diagram holds.*



2.3.2 Normal Forms and Typed Lambda Calculi

The Church-Rosser property gives us some guidance in determining β-equivalence by allowing us to consider only β-reduction. A general method for determining β-equivalence is still incomplete since the process of β-reduction can go on indefinitely, such as for $((\lambda x.x\ x)\ (\lambda x.x\ x))$. Thus we are often interested in a subset of lambda terms for which β-contraction is no longer applicable. This is the content of the following definition and theorem.

Definition 2.3.1 (β-normal form)**.** *A lambda term is in β-normal form if it has no β-redexes. If t_1 and t_2 are lambda terms such that $t_1 \rhd_\beta^* t_2$ and t_2 is in β-normal form, then we say that t_2 is the β-normal form of t_1. When there is no ambiguity, we may call this simply the normal form of t_1.*

Theorem 2.3.3. *β-normal forms are unique.*

An even stronger property than having a β-normal form is for a term to be strongly β-normalizing. This means that any sequence of β-reductions is terminating and therefore reaches the β-normal form. When terms are strongly β-normalizing, we can determine β-equivalence by reducing to β-normal form and comparing for equality. Thus, when dealing with strongly β-normalizing terms, we have a complete decision procedure.

One system for ensuring terms are strongly β-normalizing is the simply typed lambda calculus [Chu40]. In this calculus we have a set of types which we assign the constants in the language. In addition, we annotate all lambdas with a type which represents the valid argument type of the function. Using this, we can build up and assign types to a whole range of terms. To begin with, we define the types in our language.

Definition 2.3.2 (Simple types)**.** *Simple types are defined by*

$$T ::= \alpha \mid T \to T$$

where α ranges over a nonempty set of base types.

We use the names A and B to represent types. We assign a type A to a term t by creating a typing judgement which says that A is a valid type for t. The form of this judgement is $\Gamma \vdash_\Sigma t : A$ where Γ, called the context, contains type assignments for the free variables and Σ, called the signature, contains type assignments for the constants. The context maintains type assignments for free variables using a stack of types $A_1.A_2.\cdots.A_n.\emptyset$ which represent types for the free variables $\#1$, $\#2$, $\ldots$, $\#n$, respectively. The rules for constructing these typing judgements are given in Figure 2.1. Notice that in these rules, the lambdas are annotated with the type of their arguments.

$$\frac{}{\Gamma \vdash_\Sigma c : A} \text{ where } c : A \in \Sigma \qquad \frac{}{A.\Gamma \vdash_\Sigma \#1 : A}$$

$$\frac{\Gamma \vdash_\Sigma \#(i-1) : A'}{A.\Gamma \vdash_\Sigma \#i : A'} \text{ where } i > 1 \qquad \frac{\Gamma \vdash_\Sigma t_1 : B \to A \quad \Gamma \vdash_\Sigma t_2 : B}{\Gamma \vdash_\Sigma (t_1\ t_2) : A}$$

$$\frac{A.\Gamma \vdash_\Sigma t : B}{\Gamma \vdash_\Sigma (\lambda_A\, t) : A \to B}$$

Figure 2.1: Typing rules for the simply typed lambda calculus

The two key theorems about the simply typed lambda calculus are that β-reduction preserves typing and that typed terms are strongly β-normalizing.

Theorem 2.3.4 (Preservation of types)**.** *If* $\Gamma \vdash_\Sigma t : A$ *and* $t \rhd_\beta t'$ *then* $\Gamma \vdash_\Sigma t' : A$*.*

Theorem 2.3.5 (Strong normalization of typed terms)**.** *If* $\Gamma \vdash_\Sigma t : A$ *then* t *is strongly* β*-normalizing.*

The simply typed lambda calculus is only one example of a typing system which may be layered on top of the lambda calculus. Many other typing systems have been developed and put to use in real systems. For instance, λProlog [NM91] uses a polymorphic simply typed lambda calculus [NP92] and Twelf [PS99] uses the a dependently typed lambda calculus [HHP93].

2.4 Existential Variables and Substitution

The representational use of the lambda calculus produces a need for variables which can be instantiated. For example, suppose we want to express a rewrite rule in logic which says $\forall x.(F \wedge G(x)) \to F \wedge \forall x.G(x)$. Such a rule allows us to pull a term out of a universal quantifier if it does not contain the variable being quantified over. Using the ideas discussed in the introduction of this thesis we can think of encoding the left-hand side of this rule as $forall\ (\lambda x.F \wedge (G\ x))$ where F and G are variables which will be instantiated upon matching this rule with a specific instance. These variables are called *meta variables* and determining appropriate substitutions for them is called higher-order unification.

The logical interpretation of meta variables requires that substitutions cannot capture variables. Thus in our example above, if F or G contain free occurrences of x, then the bound variable x must be renamed before the substitutions are performed. This restriction is actually a benefit to the use of meta variables because it allows us to have precise control over the possible variable occurrences within a substitution. For instance, although G cannot bind the variable x in our example, x is provided as an argument so that G can be of the form $(\lambda x.G')$ where G' is a term which has free occurrences of x. The process of β-contraction then ties the knot and associates the free occurrences of x in G' with the argument x in $(G\ x)$. For example, consider substituting $(\lambda x.x + x > x)$ for G. Then the left-hand side becomes

$$forall\ (\lambda x.F \wedge ((\lambda x.x + x > x)\ x)) \rhd_\beta forall\ (\lambda x.F \wedge (x + x > x))$$

Conversely, since x is not an argument to F we know that F does not depend on x and so moving the F outside of the quantifier $\forall x$ is a logically sound operation.

Chapter 3

The Suspension Calculus

The lambda calculus revolves around β-contraction. In turn, β-contraction depends on a monolithic substitution operation which is impractical for use in actual implementations. A solution to this is to move the substitution operation down from the meta level into the object level, *i.e.*, make it explicitly part of the syntax of the lambda calculus. This allows for direct manipulation of substitutions and therefore fine-grained control over the substitution process. The focus of this chapter is on a specific explicit substitution calculus: the suspension calculus.

The first half of this chapter, extending from Section 3.1 to Section 3.6, introduces the suspension calculus and possible variations on it. We start this introduction by motivating the notation used to encode substitutions in the suspension calculus and defining notation and rules based on this motivation. We then describe the relationship between the suspension calculus of this thesis and the original suspension calculus from which it is created. Next we show how the suspension calculus admits typing in the style of the simply-typed lambda calculus. We then discuss different ways in which meta variables can be added to the calculus.

The second half of this chapter, comprising the remaining sections, deals with proving important properties of the suspension calculus. First, we prove that the rules governing substitution are terminating, thus reflecting the finite nature of substitution in the lambda calculus. Second, we show that these substitution rules are confluent, thus the choices we make in performing substitution always result in the same normal form. Third, we show that the suspension calculus faithfully models the process of β-reduction in the lambda calculus. Fourth, we prove that the property of confluence extends to the full suspension calculus, thus making it a candidate for new approaches to unification. Finally, we prove a property intrinsic to the suspension calculus which relates different ways of representing substitutions.

3.1 Motivation for the Encoding of Substitutions

Before we formally and explicitly define the syntax of the suspension calculus, it is beneficial to consider what information needs to be reflected into the syntax. Note that we are doing this reflection in the context of the de Bruijn notation since it provides a unique representation of α-equivalent terms. The first difficultly in this respect is that substitution in the de Bruijn notation is an operation with infinitely many arguments. For instance we have the β-contraction rule which say

$$((\lambda\, t_1)\ t_2) \rhd_\beta S(t_1; t_2, \#1, \#2, \ldots).$$

The substitution here is for infinitely many variables and thus no naive embedding of this into the syntax will work. Instead, it helps to start with a fresh view of the substitution needed for β-reduction.

Consider the following term for which we want to perform β-reduction but delay its effect on t,

$$(\ldots((\lambda \ldots(\lambda \ldots((\lambda \ldots t \ldots)\ s_1)\ldots)\ldots)\ s_2)\ldots) \tag{3.1}$$

Here we have a redex with s_2 as an argument and within the body of this redex we have another redex with s_1 as an argument. We want to consider the effect on t of contracting these redexes. That is, we wish to produce a term of the following form:

$$(\ldots(\ldots(\lambda \ldots(\ldots t' \ldots)\ldots)\ldots)\ldots) \tag{3.2}$$

where t' is an encoding of the term t together with the information needed to perform the substitutions generated by contracting the two redexes. We call t' a *suspension* since it represents a suspended substitution. The information in this suspension consists of substitutions for some variables and renumberings for the other variables. In order to express this information, it helps to distinguish between two types of variables in t: those that referred to a variable bound within the outermost lambda that is contracted and those that are free with respect to the outermost lambda. We use the outermost lambda as the reference point for this information since we hope to make all information local to the contractions being made, *i.e.*, the context of our redexes should not affect the term t'. In the discussion that follows, we shall fixate on the term starting with the outermost lambda that is contracted and ignore the context in which it occurs.

For the variables that are free there is a renumbering which must be done to account for the lambdas which occurred before the reduction that are gone after the reduction. In order to define this, we introduction the idea of an *embedding level* which is the depth of a term counted by the number of lambdas between it and the top level. The embedding level of t in (3.1) is called the *old embedding level*, ol, and the embedding level of t' in (3.2) is called the *new embedding level*, nl. For example, if the lambdas shown are the only lambdas in the term, then old embedding level is 3 and the new embedding level is 1. The number of lambdas that are removed is $ol - nl$ and thus this is the renumbering to be done on the free variables. Furthermore, we can easily find the free variables since they are the ones whose index is greater than their embedding level, *e.g.*, if #3 occurs at embedding level 2 then we know it occurs underneath two lambdas and represents the first free variable outside of these lambdas.

In addition to renumberings for the free variables, we must provide substitutions for the bound variables. In the case that a bound variable does not need a substitution (since its binding lambda is not contracted) we will create a dummy substitution which substitutes the first free variable, #1, thus preserving the term. Now, the terms within substitutions often come from a different embedding level than where we are thinking of substituting them and must be renumbered so that their free variables are not captured by the context into which we substitute. In order to do this, for each substitution we keep a number indicating the embedding level from which it came, say l. Then when we need to perform a substitution we increment all free variables in the substituted term by $nl - l$. We keep these substitutions together with their embedding levels in a list, ordered from the first bound variable to the last. This allows us to simply add a dummy substitution to the front of this list in order to shift all indices when we descend underneath an abstraction.

Given the previous information for encoding substitutions, we write our encoding as $[\![t, ol, nl, e]\!]$ where t is the term over which substitution is performed, ol is the old embedding level, nl is the new embedding level, and e, called the environment, is a list of substitutions for the bound variables. The overall term $[\![t, ol, nl, e]\!]$ is called a suspension. If we looked further into e, its structure would be $(t_1, l_1) :: (t_2, l_2) :: \ldots :: nil$ where the t_i are terms and l_i are the corresponding embedding levels.

The general operation of the suspension calculus will be to perform β-contractions which produce suspensions and then to apply rewriting rules which move these suspensions deeper and deeper into the tree until they are applied to a final term. During this, it is natural that we might encounter a term of the form $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!]$, which represents a term t together with two substitutions to be performed on it sequentially. We can think of merging these two substitutions to produce a term of the form $[\![t, ol', nl', e']\!]$, *i.e.*, a single suspension which encodes the information of both of the previous ones. This requires careful consideration of the values for ol' and nl' together with new syntax to represent the shape of e'.

We can determine values for ol' and nl' by thinking carefully about the embedding levels in the term $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!]$. Here we have two substitutions over t which possibly overlap with each other. In detail, the effect on t is first to substitute for the first ol_1 free variables and then to raise all the free variables by nl_1. This raising has the effect of embedding the term within nl_1 abstractions. Then the second substitution walks over the resulting term, substituting for the first ol_2 free variables and then raising all the free variables by nl_2. Note that some of these ol_2 substitutions will be vacuous since the free variables have all been raised by nl_1. In fact, if $nl_1 \geq ol_2$, then all of the substitutions of the second suspension are vacuous. In this case we have $ol' = ol_1$ since only the first ol_1 substitutions are performed. Also, $nl' = nl_2 + (nl_1 - ol_2)$ since the raising done is that of nl_2 and nl_1 minus the ol_2 vacuous substitutions. In the case when $nl_1 < ol_2$, we have $ol' = ol_1 + (ol_2 - nl_1)$ since we have the ol_1 substitutions and all but the first nl_1 of the ol_2 substitutions. Also, $nl' = nl_2$ since all of the nl_1 raisings are consumed by the ol_2 substitutions and so the only raising left over is from the second suspension. These two branching cases for ol' and nl' can be coalesced by using the minus operator on natural numbers. Then we have $ol' = ol_1 + (ol_2 \dot{-} nl_1)$ and $nl' = nl_2 + (nl_2 \dot{-} ol_1)$ in both cases.

We must also determine the shape of e' after merging $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!]$ into $[\![t, ol', nl', e']\!]$. The result should roughly be the substitutions of e_1, modified by the substitutions in e_2, together with some tail portion of e_2. For each term in e_1, we can use nl_1 to compute the number of abstractions in which it is embedded. Using this we can prune off the first elements of e_2 which correspond to these abstractions. When we have done this for all elements of e_1, we only have left to determine which tail portion of e_2 to include. The length of this tail portion should be the number of abstractions consumed by the second suspension, ol_2, minus the number of abstractions created by the first suspension, nl_1. Thus we can compute the total shape of e' by knowing only e_1, e_2, ol_2, and nl_1. We write the resulting form as $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ and we call this a *merged environment*.

3.2 Syntax of the Suspension Calculus

In this section we formally define the syntax of the suspension calculus and present measures for assessing the wellformedness of expressions in this calculus. We begin by defining the "pre-syntax" of suspension expressions, which is the raw syntax without any constraints on wellformedness.

Definition 3.2.1 (Pre-syntax of suspension expressions)**.** *The "pre-syntax" of suspension expressions is given by the following definitions of the syntactic categories of "terms" and "environments."*

$$\begin{array}{lcl} t & ::= & c \mid \#i \mid (t\ t) \mid (\lambda\, t) \mid [\![t, n, n, e]\!] \\ e & ::= & nil \mid ((t, l) :: e) \mid \{\!\{e, n, n, e\}\!\} \end{array}$$

Here c ranges over an enumerable set of constants, i ranges over the natural numbers, and n and l ranges over the non-negative integers.

We call $\#i$ a variable index or reference, and we call $(t\ t)$ and $(\lambda\, t)$ application and abstraction, respectively. The term $[\![t, n, n, e]\!]$ is called a *suspension*. The operation $::$ is a consing operator on lists and the (t, l) component of $((t, l) :: e)$ is called an *environment term*. Finally, $\{\!\{e, n, n, e\}\!\}$ is called a *merged environment*. We collectively refer to all terms, environments, and environment terms generated by the above definition as *suspension expressions*. We will drop parenthesis by assuming application is left associative, the scope of a lambda extends as far right as possible, and $::$ is right associative. We will also assume the obvious definition of subexpression.

In order to move from pre-syntax to syntax, we need to consider constraints on expressions so that they "make sense." For example, in a term of the form $[\![t, ol_1, nl_1, e_1]\!]$ we are thinking of performing substitution for the first ol_1 free variables using the substitutions in e_1. Thus the number of substitutions in e_1, called the length of e_1, must be ol_1. Also, for each substitution in e_1 we include an embedding level from which the substitution came. Since substitutions will always move inward, it must be that we keep moving to deeper embedding levels. Thus our current embedding level, nl_1, must be greater than or equal to the embedding level of every environment term in e_1. We enforce this by defining a measure called the level of e_1. These measures are the content of the following definitions.

Definition 3.2.2 (Length of an environment)**.** *The length of an environment e is denoted by $len(e)$ and is defined recursively by*

$$\begin{aligned} len(nil) &= 0 \\ len((t, n) :: e) &= 1 + len(e) \\ len(\{\!\{e_1, nl_1, ol_2, e_2\}\!\}) &= len(e_1) + (ol_2 \dot{-} nl_1) \end{aligned}$$

Definition 3.2.3 (Level of an environment or environment term)**.** *The level of an environment e is denote $lev(e)$ and is defined recursively by*

$$lev(nil) = 0$$
$$lev((t,n) :: e) = n$$
$$lev(\{\!\{e_1, nl_1, ol_2, e_2\}\!\}) = lev(e_2) + (nl_1 \dot{-} ol_2)$$

Given these two definitions, we can define the syntax of suspension expressions

Definition 3.2.4 (Syntax of suspension expressions)**.** *The syntax of suspension expressions is those expressions in Definition 3.2.1 with the following additional wellformedness constraints,*

1. *In any subexpression of the form $[\![t, ol, nl, e]\!]$, we must have $len(e) = ol$ and $lev(e) \leq nl$.*
2. *In any subexpression of the form $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$, we must have $len(e_2) = ol_2$ and $lev(e_1) \leq nl_1$.*
3. *In any subexpression of the form $(t, n) :: e$, we must have $lev(e) \leq n$.*

3.3 Rules of the Suspension Calculus

Now that we have created syntax which embeds substitutions we can consider rules which operate on this modified syntax. The effect of these rules should be in three parts: (1) to contract beta redexes to produce suspensions, (2) to move these suspensions down in the syntax tree until they can be applied, and (3) to merge suspensions and compute the resulting merged environment. The complete rules of the suspension calculus are listed in Figure 3.1 and are divided into three categories described above: the β_s rules, the reading rules, and the merging rules.

The β_s rule simulates β-contraction in the suspension calculus using the suspension syntax to encode the effect of substitution. This rule rewrites the β-redex $((\lambda\, t_1)\ t_2)$ to $[\![t_1, 1, 0, (t_2, 0) :: nil]\!]$ which says that we substitute t_2 in for the first free variable in t_1 and decrement all other free variables by one.

The second category of rules is the reading rules, (r1)-(r6), which provide a means for moving suspensions down in a term and also performing substitutions. Taken together, the

(β_s) $((\lambda\, t_1)\ t_2) \rightarrow [\![t_1, 1, 0, (t_2, 0) :: nil]\!]$.

(r1) $[\![c, ol, nl, e]\!] \rightarrow c$, provided c is a constant.

(r2) $[\![\#i, 0, nl, nil]\!] \rightarrow \#j$, where $j = i + nl$.

(r3) $[\![\#1, ol, nl, (t, l) :: e]\!] \rightarrow [\![t, 0, nl', nil]\!]$, where $nl' = nl - l$.

(r4) $[\![\#i, ol, nl, (t, l) :: e]\!] \rightarrow [\![\#i', ol', nl, e]\!]$,
where $i' = i - 1$ and $ol' = ol - 1$, provided $i > 1$.

(r5) $[\![(t_1\ t_2), ol, nl, e]\!] \rightarrow ([\![t_1, ol, nl, e]\!]\ [\![t_2, ol, nl, e]\!])$.

(r6) $[\![(\lambda\, t), ol, nl, e]\!] \rightarrow (\lambda\, [\![t, ol', nl', (\#1, nl') :: e]\!])$,
where $ol' = ol + 1$ and $nl' = nl + 1$.

(m1) $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!] \rightarrow [\![t, ol', nl', \{\!\{e_1, nl_1, ol_2, e_2\}\!\}]\!]$,
where $ol' = ol_1 + (ol_2 \dot{-} nl_1)$ and $nl' = nl_2 + (nl_1 \dot{-} ol_2)$.

(m2) $\{\!\{e_1, nl_1, 0, nil\}\!\} \rightarrow e_1$.

(m3) $\{\!\{nil, 0, ol_2, e_2\}\!\} \rightarrow e_2$.

(m4) $\{\!\{nil, nl_1, ol_2, (t, l) :: e_2\}\!\} \rightarrow \{\!\{nil, nl_1', ol_2', e_2\}\!\}$,
where $nl_1' = nl_1 - 1$ and $ol_2' = ol_2 - 1$, provided $nl_1 \geq 1$.

(m5) $\{\!\{(t, n) :: e_1, nl_1, ol_2, (s, l) :: e_2\}\!\} \rightarrow \{\!\{(t, n) :: e_1, nl_1', ol_2', e_2\}\!\}$,
where $nl_1' = nl_1 - 1$ and $ol_2' = ol_2 - 1$, provided $nl_1 > n$.

(m6) $\{\!\{(t, n) :: e_1, n, ol_2, (s, l) :: e_2\}\!\} \rightarrow ([\![t, ol_2, l, (s, l) :: e_2]\!], m) :: \{\!\{e_1, n, ol_2, (s, l) :: e_2\}\!\}$,
where $m = l + (n \dot{-} ol_2)$.

Figure 3.1: Rewrite rules for the suspension calculus

β_s rule and the reading rules form an adequate simulation of the de Bruijn calculus, as we shall prove in Section 3.9.

The merging rules, (m1)-(m6), are the final category of rules. The (m1) rule enables us to merge two suspension into a single suspension, at the cost of creating a merged environment. The rules (m2)-(m6) then allow us to evaluate this merged environment in a lazy way.

Definition 3.3.1. *The reduction relations generated by the rules in Figure 3.1 are denoted by* $\rhd_{\beta_s}$*,* $\rhd_r$*, and* $\rhd_m$*. The relations* $\rhd_{rm}$*,* $\rhd_{r\beta_s}$*, and* $\rhd_{rm\beta_s}$ *are the appropriate unions of those relations. If* R *corresponds to any of these relations then we will use* R^* *to denote its reflexive and transitive closure.*

The following example illustrates a use of these rules where t, s_1, and s_2 are arbitrary suspension expressions. This example is a simplified version of (3.1).

$$
\begin{aligned}
&(\lambda\,\lambda\,((\lambda\,t)\ s_1))\ s_2\\
&\rhd^*_{\beta_s} [\![\lambda\,[\![t,1,0,(s_1,0) :: nil]\!],1,0,(s_2,0) :: nil]\!]\\
&\rhd_r \lambda\,[\![[\![t,1,0,(s_1,0) :: nil]\!],2,1,(\#1,1) :: (s_2,0) :: nil]\!]\\
&\rhd_m \lambda\,[\![t,3,1,\{\!\{(s_1,0) :: nil,0,2,(\#1,1) :: (s_2,0) :: nil\}\!\}]\!]\\
&\rhd^*_m \lambda\,[\![t,3,1,([\![s_1,2,1,(\#1,1) :: (s_2,0) :: nil]\!],1) :: (\#1,1) :: (s_2,0) :: nil]\!]
\end{aligned}
$$

The outermost suspension here encodes three substitutions to be made over t. The first substitution is s_1, modified by substituting s_2 for its second free variable. The second substitution is a dummy substitution which corresponds to the lambda that remains after contraction. Finally, the last substitution corresponds to substituting in s_2 and since the embedding level of this is one less than the new embedding level, we will have to raise all the free variables of s_2 which corresponds to our substituting of s_2 underneath a lambda.

In order for our rules to make sense, they need to produce terms that make sense. In formal terms, we need to ensure that using a rule on a well-formed expression produces a well-formed expression.

Theorem 3.3.1. *Let e be a well-formed suspension expression and let $e \rhd_{rm\beta_s} e'$. Then e' is a well-formed suspension expression.*

Proof. This property must be proved simultaneously with two other properties: if e is an environment then $lev(e) \geq lev(e')$ and $len(e) = len(e')$. The reason is that if we could rewrite an environment so that its level increases or so its length changes, then we might break the wellformedness of an expression such as $[\![t, ol, nl, e]\!]$ where $lev(e) \leq nl$ and $len(e) = ol$ by rewriting e to e' such that $lev(e') > nl$ or $len(e) \neq ol$. The proof of all three properties simultaneously is a simple case analysis on the rewrite from e to e'. We give two examples here in order to give the flavor of the argument.

Consider (r3) and suppose that the left hand side is well-formed. This means that $ol = len((t,l) :: e)$ and $nl \geq lev((t,l) :: e) = l$. In order for the right hand side to be well-formed we must have $0 = len(nil)$ and $nl' \geq lev(nil) = 0$. The first is trivially true, and the second requires that we show $nl - l \geq 0$ which follows from $nl \geq l$.

As a second example, consider (m6). Assuming the left hand side is well formed yields $n \geq lev((t,n) :: e_1)$, $ol_2 = len((s,l) :: e_2)$, and $lev(et) \geq lev(e)$. In order to show the right hand side is well-formed we must have $l \geq lev((s,l) :: e_2)$, $ol_2 = len((s,l) :: e_2)$, $lev(et) \geq lev((s,l) :: e_2)$, $m \geq lev(\{\!\{e_1, n, ol_2, (s,l) :: e_2\}\!\}$, $n \geq lev(e_1)$, $ol_2 = len((s,l) :: e_2)$, and $l \geq lev(e_2)$. All of these follow directly. Since (m6) is a rule on environments, we must also verify that this rewriting does not cause the level of this environment to increase. The level of the left hand side is $lev((s, l) :: e_2) + (n \dot{-} ol_2)$ and the level of the right hand side is $m = l + (n \dot{-} ol_2)$. Since $l = lev((s, l) :: e_2)$, this follows easily. Finally, it is plain to see that the length is preserved. □

From here on we will speak only of well-formed suspension expressions and thus drop the "well-formed" label on them. The final definition and lemma of this section are ones that allows us to massage environments into a convenient form for use with the rewrite rules. This will be of importance in Section 3.4 and Section 3.8.

Definition 3.3.2 (Simple environments and truncating)**.** *A simple environment is one of the form* $(t_0, l_0) :: (t_1, l_1) :: \ldots :: (t_{n-1}, l_{n-1}) :: nil$*, with* n *possibly zero. For* $0 \leq i < n$*, we write* $e\{i\}$ *to denote the truncated environment with the first* i *elements removed,* i.e., $(t_i, l_i) :: \ldots :: (t_{n-1}, l_{n-1}) :: nil$*, with* i *possibly zero. We extend this notation by letting* $e\{i\}$ *denote* nil *in the case that* $i \geq len(e)$ *for any simple environment* e*.*

Lemma 3.3.1. *Let* e *be an environment. Then there exists a simple environment* e' *such that* $e \rhd_m^* e'$*.*

Proof. The proof is by case analysis on the structure of e, basically showing that if e is not a simple environment then we can always apply a rule (m2)-(m6) to it. Connecting this to the result requires that we also know that the rewrite rules are terminating, which is shown in Section 3.7. □

3.4 Relationship to the Original Suspension Calculus

The current suspension calculus is based on the original suspension calculus by Nadathur and Wilson [NW98]. There are two differences between these calculi: the way dummy substitutions are handled in the rule (r6) and the way merging is performed using (m2)-(m6). In this section we highlight these differences and explain the connection between these two calculi. Before we begin, however, we note that the original suspension calculus had a separate syntactic class for environments terms because it had more possibilities for such expressions. Hence, in this section we will treat environment terms as a separate syntactic class in both calculi.

The first difference is how dummy substitutions are handled in the case of rule (r6). When pushing a suspension underneath an abstraction there is a need to generate a substitution for the bound variable. Since we are not actually contracting a redex, this substitution should have no net effect and is thus a dummy substitution compared to those substitutions generated by β-contraction. In the original calculus, the rule for pushing a suspension underneath an abstraction had the form

$$[\![\lambda\, t, ol, nl, e]\!] \rightarrow \lambda\, [\![t, ol + 1, nl + 1, @nl :: e]\!]$$

This new $@nl$ environment term was conceived of as an optimization to separate real and dummy substitutions, and it also has the benefit of simplifying the proof of termination for the reading and merging rules. Nevertheless, we can simplify the calculus significantly by using the environment term $(\#1, nl+1)$ instead. This environment term has the same effect the same effect and also allows us to have a simpler system since we can exclude all the old rules for manipulating $@nl$ forms. With this change, the results of the original suspension calculus paper still hold, which we will assume [NW98].

The larger difference between the two calculi is the way merging is performed using (m2)-(m6). Taking into account the change above, the β_s and reading rules are the same in both calculi, but the merging rules are still significantly different. The merging rules for the original calculus are presented in Figure 3.2. The rule (m8') makes use of a measure called the index which we will say more about later in this section. For now, it is sufficient to think of the index as a measure very similar to the level. Pay special attention to (m5') which says

$$\{\!\{et :: e_1, nl_1, ol_2, e_2\}\!\} \rightarrow \langle\!\langle et, nl_1, ol_2, e_2\rangle\!\rangle :: \{\!\{e_1, nl_1, ol_2, e_2\}\!\}$$

The effect of this rule is to eagerly propagate the effects of e_2 onto the environment term et by creating a new environment term $\langle\!\langle et, nl_1, ol_2, e_2\rangle\!\rangle$. This new form is then used to prune e_2 using rules (m6') and (m7') until only the portion relevant to et was left. In the current calculus, this pruning is done using the form $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ using rule (m5) and thus the work of this pruning is shared for each environment term of e_1. This change in the current calculus allows us to use fewer syntactic forms and also fewer rules, both of which reduce the energy required to understand the calculus.

A possible downside of these simplifications is that we might lose some desirable theoretical properties of the calculus. This turns out not to be the case as we prove later in this chapter. In fact, the simplifications to the calculus allow new results such as a typed version of the calculus in Section 3.5 and a translation to another explicit substitution calculus in

(m1') $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!] \rightarrow [\![t, ol', nl', \{\!\{e_1, nl_1, ol_2, e_2\}\!\}]\!]$,
where $ol' = ol_1 + (ol_2 \dot{-} nl_1)$ and $nl' = nl_2 + (nl_1 \dot{-} ol_2)$.

(m2') $\{\!\{nil, nl, 0, nil\}\!\} \rightarrow nil$.

(m3') $\{\!\{nil, 0, ol, e\}\!\} \rightarrow e$.

(m4') $\{\!\{nil, nl, ol, et :: e\}\!\} \rightarrow \{\!\{nil, nl', ol', e\}\!\}$,
where $nl, ol \geq 1$, $nl' = nl - 1$ and $ol' = ol - 1$.

(m5') $\{\!\{et :: e_1, nl, ol, e_2\}\!\} \rightarrow \langle\!\langle et, nl, ol, e_2\rangle\!\rangle :: \{\!\{e_1, nl, ol, e_2\}\!\}$.

(m6') $\langle\!\langle et, nl, 0, nil\rangle\!\rangle \rightarrow et$.

(m7') $\langle\!\langle et, nl, ol, et' :: e\rangle\!\rangle \rightarrow \langle\!\langle et, nl', ol', e\rangle\!\rangle$,
where $nl' = nl - 1$ and $ol' = ol - 1$, provided $ind(et) < nl$.

(m8') $\langle\!\langle (t, nl), nl, ol, et :: e\rangle\!\rangle \rightarrow ([\![t, ol, l', et :: e]\!], m)$
where $l' = ind(et)$ and $m = l' + (nl \dot{-} ol)$.

Figure 3.2: Merging rules for the original suspension calculus

Section 4.2. In the rest of this section, we will look at a more formal relationships between our calculus and its predecessor.

Suspension expressions in our context are a subset of the original suspension expressions. The only difficultly in showing this is that wellformedness rules for the original suspension calculus are defined the same as in Definition 3.2.4 except with the measure level replaced by the measure index, defined as follows.

Definition 3.4.1 (Index of an environment or environment term). *Given a natural number i, the i-th index of an environment e is denoted by $ind_i(e)$ and is defined as follows:*

1. *If e is nil then $ind_i(e) = 0$.*

2. *If e is $(t, k) :: e'$ then $ind_i(e)$ is k if $i = 0$ and $ind_{i-1}(e')$ otherwise.*

3. *If e is $\{\!\{e_1, nl, ol, e_2\}\!\}$, let $m = (nl \dot{-} ind_i(e_1))$ and $l = len(e_1)$. Then*

$$ind_i(e) = \begin{cases} ind_m(e_2) + (nl \dot{-} ol) & \text{if } i < l \text{ and } len(e_2) > m \\ ind_i(e_1) & \text{if } i < l \text{ and } len(e_2) \leq m \\ ind_{(i-l+nl)}(e_2) & \text{if } i \geq l. \end{cases}$$

The index of an environment, denoted by $ind(e)$, is $ind_0(e)$.

Intuitively, the index of an environment is the embedding level of its first environment term and zero if the environment is nil. The index of an environment term et is the value of n for which $et \rhd_m^* (t, n)$. We can consider the index measure on expressions in the current calculus as well. Note that this measure is equal to the level on environments terms in the current calculus because they already have the form (t, n). The only difference in the current calculus between index and level is in the case of merged environments, where the level is an upper bound on the index.

Lemma 3.4.1. *Let e be a well-formed environment in the current calculus. Then $lev(e) \geq ind(e)$.*

Proof. First generalize to $lev(e) \geq ind_i(e)$ for all i. Then the proof proceeds by induction on the structure of e. □

Lemma 3.4.2. *Well-formed terms in the current suspension calculus are well-formed in the original suspension calculus.*

Proof. This is a direct consequence of the previous lemma. For example, if $[\![t, ol, nl, e]\!]$ is a well-formed term of the current calculus then we have $ol = len(e)$ and $nl \geq lev(e) \geq ind(e)$. Thus it is a well-formed term in the original suspension calculus. □

We now know that the well-formed expressions we are working with in the current calculus are also well-formed in the original calculus. We can also show that the rules of the current calculus are either derived or admissible rules for the original calculus. We will consider each rule of the current calculus in turn, ignoring rules that are the same in both calculi, *i.e.*, (m1), (m3), and (m4).

The rule (m2) operates on an environment of the form $\{\!\{e_1, nl_1, 0, nil\}\!\}$. By Lemma 3.3.1, the environment e_1 can be rewritten to the form $et_0 :: et_1 :: \ldots :: et_{n-1} :: nil$. Then by applying (m5') n times we can get

$$\langle\!\langle et_0, nl_1, 0, nil\rangle\!\rangle :: \langle\!\langle et_1, nl_1, 0, nil\rangle\!\rangle :: \ldots :: \langle\!\langle et_{n-1}, nl_1, 0, nil\rangle\!\rangle :: \{\!\{nil, nl_1, 0, nil\}\!\}$$

Using (m2') and n applications of (m6') this rewrites to $et_0 :: et_1 :: \ldots :: et_{n-1} :: nil$. Thus both $\{\!\{e_1, nl_1, 0, nil\}\!\}$ and e_1 can rewrite to a common term and therefore the rule (m2) is admissible.

For rule (m5) we cite the original suspension paper where Lemma 6.10 states that $\{\!\{e_1, nl + 1, ol + 1, et :: e_2\}\!\}$ and $\{\!\{e_1, nl, ol, e_2\}\!\}$ rewrite to a common term if $ind(e_1) \leq nl$

[NW98]. Restricting this to the case where $e_1 = (t, n) :: e'_1$ and restating the requirement as $n < nl + 1$ yields the rule (m5). Thus (m5) is admissible.

Finally consider (m6). Assuming $\{\!\{(t, n) :: e_1, n, ol_2, et :: e_2\}\!\}$ is a term in the current suspension calculus, we can rewrite it using (m5') followed by (m8') to produce

$$([\![t, ol_2, l, et :: e_2]\!], m) :: \{\!\{e_1, n, ol_2, et :: e_2\}\!\}$$

where $l = ind(et)$ and $m = l + (n \dot{-} ol_2)$. Since et is an environment term in the current calculus we have $l = lev(et)$ and thus (m6) is a derived rule of the original calculus.

We can formalize our observations into the following theorem which will be useful in Section 3.9 when we show that the current calculus properly simulates β-contraction.

Theorem 3.4.1 (Same normal form)**.** *Let x be a well-formed (current) suspension expression. Then the $\rhd_{rm}$-normal form of x is also the $\rhd_{rm'}$-normal form of x.*

Proof. This theorem depends on result that the reading and merging rules are confluent and terminating in both calculi (See Section 3.7 and Section 3.8 for the current calculus, [NW98] for the original calculus). The result then follows by induction on the rewrite sequence which takes x to its $\rhd_{rm}$-normal form. □

3.5 Typed Version of the Suspension Calculus

In this section we present a typed version of the suspension calculus which was not previously possible with the original suspension calculus. This typed version is is consistent with the simply-typed lambda calculus from Section 2.3.2 and motivated in the same way. In fact our typing judgment for terms of the form c, $(\lambda\, t)$, and $(t_1\ t_2)$ is the same as in the simply-typed lambda calculus, but significant complexity is introduce to handle the new judgment for $[\![t, ol, nl, e]\!]$. The issue is that we must interpret the term t in the context of the substitutions encoded in e and relative to nl. This necessitates a new judgment which talks about the effect of an environment (relative to an embedding level) on a typing context. The form of this judgment is $\Gamma \vdash_\Sigma e \rhd_{nl} \Gamma'$. Where Γ and Γ' are contexts, Σ is a signature, e is an environment, and nl is an integer for the embedding level. All of typing rules are presented in Figure 3.3.

The following result establish the wellformedness of our typing judgments.

Theorem 3.5.1. *Type judgments are preserved by the rewrite relations*

$$\frac{}{\Gamma \vdash_\Sigma c : A} \text{ where } c : A \in \Sigma \qquad \frac{}{A.\Gamma \vdash_\Sigma \#1 : A}$$

$$\frac{\Gamma \vdash_\Sigma \#(i-1) : A'}{A.\Gamma \vdash_\Sigma \#i : A'} \text{ where } i > 1 \qquad \frac{\Gamma \vdash_\Sigma t_1 : B \to A \quad \Gamma \vdash_\Sigma t_2 : B}{\Gamma \vdash_\Sigma (t_1\ t_2) : A}$$

$$\frac{A.\Gamma \vdash_\Sigma t : B}{\Gamma \vdash_\Sigma (\lambda_A\, t) : A \to B} \qquad \frac{\Gamma \vdash_\Sigma e \rhd_{nl} \Gamma' \quad \Gamma' \vdash_\Sigma t : A}{\Gamma \vdash_\Sigma [\![t, ol, nl, e]\!] : A}$$

$$\frac{}{\Gamma \vdash_\Sigma nil \rhd_0 \Gamma} \qquad \frac{\Gamma \vdash_\Sigma nil \rhd_{nl-1} \Gamma'}{A.\Gamma \vdash_\Sigma nil \rhd_{nl} \Gamma'} \text{ where } nl > 0$$

$$\frac{\Gamma \vdash_\Sigma (t,n) :: e \rhd_{nl-1} \Gamma'}{A.\Gamma \vdash_\Sigma (t,n) :: e \rhd_{nl} \Gamma'} \text{ where } nl > n \qquad \frac{\Gamma \vdash_\Sigma t : A \quad \Gamma \vdash_\Sigma e \rhd_n \Gamma'}{\Gamma \vdash_\Sigma (t,n) :: e \rhd_n A.\Gamma'}$$

$$\frac{\Gamma \vdash_\Sigma e_2 \rhd_{nl-(nl' \dot{-} ol')} \Gamma' \quad \Gamma' \vdash_\Sigma e_1 \rhd_{nl'} \Gamma''}{\Gamma \vdash_\Sigma \{\!\{e_1, nl', ol', e_2\}\!\} \rhd_{nl} \Gamma''}$$

Figure 3.3: Typing rules for the typed suspension calculus

Proof. The critical thing to show is that if $\ell \to r$ is a rewrite rule and $\Gamma \vdash_\Sigma \ell : A$ (respectively $\Gamma \vdash_\Sigma \ell \rhd_{nl} \Gamma'$) then $\Gamma \vdash_\Sigma r : A$ (respectively $\Gamma \vdash_\Sigma r \rhd_{nl} \Gamma'$). The proof proceeds by cases on the rewrite rule and we focus here on a few interesting cases.

Let the rewrite rule be (β_s). Then we are given that $\Gamma \vdash_\Sigma ((\lambda_A\, t_1)\ t_2) : B$ from which we know the derivation tree must be

$$\frac{\dfrac{\dfrac{D_1}{A.\Gamma \vdash_\Sigma t_1 : B}}{\Gamma \vdash_\Sigma (\lambda_A\, t_1) : A \to B} \qquad \dfrac{D_2}{\Gamma \vdash_\Sigma t_2 : A}}{\Gamma \vdash_\Sigma ((\lambda_A\, t_1)\ t_2) : B}$$

Where D_1 and D_2 are appropriate derivations. We can then use these derivations to construct the following typing judgment.

$$\dfrac{\dfrac{\dfrac{D_2}{\Gamma \vdash_\Sigma t_2 : A} \qquad \Gamma \vdash_\Sigma nil \rhd_0 \Gamma}{\Gamma \vdash_\Sigma (t_2, 0) :: nil \rhd_0 A.\Gamma} \qquad \dfrac{D_1}{A.\Gamma \vdash_\Sigma t_1 : B}}{\Gamma \vdash_\Sigma [\![t_1, 1, 0, (t_2, 0) :: nil]\!]}$$

This completes the argument for (β_s).

Consider the case of (r3). Then we must have the following typing derivation.

$$\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{D_1}{\Gamma \vdash_\Sigma t : A} \qquad \dfrac{D_2}{\Gamma \vdash_\Sigma e \rhd_l \Gamma'}}{\Gamma \vdash_\Sigma (t, l) :: e \rhd_l A.\Gamma'}}{}}}{A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma (t, l) :: e \rhd_{nl-1} A.\Gamma'}}{A_{nl}.A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma (t, l) :: e \rhd_{nl} A.\Gamma'} \qquad A.\Gamma' \vdash_\Sigma \#1 : A}{A_{nl}.A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma [\![\#1, ol, nl, (t, l) :: e]\!] : A}$$

From this we can construct the typing judgment,

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_\Sigma nil \rhd_0 \Gamma}{\vdots}}{A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma nil \rhd_{nl-l-1} \Gamma}}{A_{nl}.A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma nil \rhd_{nl-l} \Gamma} \qquad \dfrac{D_1}{\Gamma \vdash_\Sigma t : A}}{A_{nl}.A_{nl-1}. \cdots .A_{l+1}.\Gamma \vdash_\Sigma [\![t, 0, nl - l, nil]\!] : A}$$

As a final example, consider the rule (m3) which yields the following typing derivation.

$$\dfrac{\dfrac{D_1}{\Gamma \vdash_\Sigma e_2 \rhd_{nl} \Gamma'} \qquad \Gamma' \vdash_\Sigma nil \rhd_0 \Gamma'}{\Gamma \vdash_\Sigma \{\!\{nil, 0, ol_2, e_2\}\!\} \rhd_{nl} \Gamma'}$$

And this directly contains the needed typing judgment.

$$\dfrac{D_1}{\Gamma \vdash_\Sigma e_2 \rhd_{nl} \Gamma'}$$

□

3.6 Meta Variables and the Suspension Calculus

The syntax of suspension expressions does not presently allow for meta variables, as described in Section 2.4. We can remedy this by adding modifying the syntax of terms to

$$t \quad ::= \quad v \mid c \mid \#i \mid (t\ t) \mid (\lambda\, t) \mid [\![t, n, n, e]\!],$$

where v represents the category of meta variables. Because such variables have a logical interpretation, any substitution for them must avoid capturing local variables. This means that any outer context cannot effect the value of such variables. To accommodate this interpretation, we can add the following to our reading rules:

(r7) $\quad [\![v, ol, nl, e]\!] \to v$, if v is a meta variable.

This rule has the same character as the (r1) rule for constants, and as a result, all the properties of the suspension calculus without logical meta variables carry over to the suspension calculus with logical meta variables.

An alternative interpretation for meta variables in one in which substitution is performed without renaming to avoid variable capture. This is referred to as the *graftable* interpretation of meta variables, and it has been found useful in higher-order unification procedures [DHK95]. The benefit of using graftable meta variables is that dependencies of meta variables on the outside context no longer need to be explicitly specified, which turns out to be a significant cost in the traditional higher-order unification algorithm [Hue75]. On the other hand, it may seem that using graftable meta variables removes any possibility of control over dependencies, but we can in fact enforce some restrictions using explicit raising. For example, to prevent the metavariable X from depending the the de Bruijn indices #1 and #2, we can replace it with the term $[\![X, 0, 2, nil]\!]$. This also us to simulate logical meta variables using graftable meta variables simply by lifting such graftable variables so that they can not depend on any of the bound variables in the term.

In order to support graftable meta variables in the calculus, we extend the syntax as before, but instead of adding the rule (r7), we leave the rules unchanged. This is consistent with the graftable interpretation: because we know nothing about such meta variables, we cannot say what effect a suspension will have on them. Adding graftable meta variables to the calculus, however, introduces some new complications with respect to confluence. For example, consider the term $((\lambda\, ((\lambda\, X)\ t_1))\ t_2)$ in which X is a graftable meta variable and t_1 and t_2 are terms in $\rhd_{rm}$-normal form. This term can be rewritten to

$$[\![[\![X, 1, 0, (t_1, 0) :: nil]\!], 1, 0, (t_2, 0) :: nil]\!]$$

and also to

$$[\![[\![X, 2, 1, (\#1, 1) :: (t_2, 0) :: nil]\!], 1, 0, ([\![t_1, 1, 0, (t_2, 0) :: nil]\!], 0) :: nil]\!],$$

amongst other terms. It is easy to see that these terms cannot now be rewritten to a common form using only the reading and (β_s) rules. The merging rules are essential to this ability and, as we see in Section 3.8, these also suffice for this purpose. Another impact of graftable meta variables is that normal forms with respect to the reading and merging rules now include the possibility of remaining suspensions, whereas without graftable meta variables such normal forms are always de Bruijn terms.

We assume henceforth that the suspension calculus includes meta variables under the graftable interpretation. For reasons already mentioned, it is easy to see that the properties we establish for the resulting calculus will hold also under the logical interpretation.

3.7 Termination of Reading and Merging Rules

The first significant property we prove for the suspension calculus is that the reading and merging rules are terminating, *i.e.*, that all $\rhd_{rm}$-sequences are finite. This is useful in all future sections because it allows us to induct on $\rhd_{rm}$-sequences, and it tells that $\rhd_{rm}$-normal forms always exist. On a deeper level, the substitution process in the lambda calculus is inherently finite and by showing that our elaboration of this substitution process is also finite, we argue convincingly for the well behaved nature of our rules.

We prove the termination of the reading and merging rules in two steps. First, we describe a collection of first-order terms and a wellfounded order on those terms using a lexicographic recursive path ordering [Der82, FZ95]. Second, we define a mapping from suspension expressions to newly described terms such that each of the reading and merging rules produces a smaller term with respect to the defined order. The desired conclusion follows from these facts. The key points of this work have been verified in Coq.[1]

We imagine the terms we describe here to be an abstract view of the suspension calculus such that only details relevant to the termination of the reading and merging rules are considered. These terms are constructed using the following (infinite) vocabulary: the following (infinite) vocabulary: the 0-ary function symbol *, the unary function symbol *lam*, and the binary function symbols *app*, *cons* and, for each positive number i, s_i. We denote this collection of terms by $\mathcal{T}$. We assume the following partial ordering $\sqsupset$ on the

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[1] http://www-users.cs.umn.edu/~agacek/pubs/gacek-masters/Termination/

signature underlying $\mathcal{T}$: $s_i \sqsupset s_j$ if $i > j$ and, for every i, $s_i \sqsupset app$, $s_i \sqsupset lam$, $s_i \sqsupset cons$ and $s_i \sqsupset *$. This ordering is now extended to the collection of terms.

Definition 3.7.1 (Term Order)**.** *The relation $\succ$ on $\mathcal{T}$ is inductively defined by the following property: Let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$; both s and t may be *,* i.e., *the number of arguments for either term may be* 0*. Then $s \succ t$ if*

1. *$f = g$ (in which case $n = m$), $(s_1, \ldots, s_n) \succ_{lex} (t_1, \ldots, t_n)$, and, $s \succ t_i$ for all i such that $1 \leq i \leq n$, or*
2. *$f \sqsupset g$ and $s \succ t_i$ for all i such that $1 \leq i \leq n$, or*
3. *$s_i = t$ or $s_i \succ t$ for some i such that $1 \leq i \leq m$.*

Here $\succ_{lex}$ denotes the lexicographic ordering induced by $\succ$.

In the terminology of [FZ95], $\succ$ is an instance of a recursive path ordering based on $\sqsupset$. It is easily seen that $\sqsupset$ is a well-founded ordering on the signature underlying $\mathcal{T}$. The results in [FZ95] then imply the following:

Lemma 3.7.1. *$\succ$ is a well-founded partial order on $\mathcal{T}$.*

We now consider the translation from suspension expressions to $\mathcal{T}$. The critical part of this mapping is the treatment of expressions of the form $[\![t, ol, nl, e]\!]$ and $\{\!\{e_1, nl, ol, e_2\}\!\}$. Because of (m1) and (m6), there is a tight relationship between the encoding of these two types of expressions. It turns out that we can ignore the differences between them when looking at our abstract view of the suspension calculus, thus we drop the *ol* and *nl* from each and encode them as s_i for some i.

To determine the appropriate value for i in s_i, we must consider how this i will be needed. We will focus on the case for $[\![t, ol, nl, e]\!]$, but the same ideas will carry over to $\{\!\{e_1, nl, ol, e_2\}\!\}$. A first attempt to translate $[\![t, ol, nl, e]\!]$ as s_i for some fixed i would fail since the rule (r3) would not yield a smaller term when applied due to the lexicographic component of our ordering. Instead, we use the the value of i as a coarse measure of the remaining substitution work, so that this value decreases in the rule (r3). In order for this to work we must count the #1 on the left-hand side of (r3) for some positive amount. This then passes the problem onto (r6) where we add a #1 to the right-hand side. In order to balance this, we assign lambdas a weight based on the number of suspensions in which they are embedded. This results in our family of measures η_j where j represents the number of levels of suspensions or merged environments that the current term is embedded underneath. In defining this measure we must be aware that the rule (m1) allows the environment e_2 to become embedded underneath an additional level. Thus the embedding level of an environment must be based on the number of suspensions in the term over which the environment applies. We call this count of suspensions the "internal embedding potential" and denote it by μ in the following definitions. In these definitions, *max* is the function that picks the larger of its two integer arguments.

Definition 3.7.2. *The measure μ that estimates the internal embedding potential of a suspension expression is defined as follows:*

1. *For a term t, $\mu(t)$ is 0 if t is a constant, a meta variable or a de Bruijn index, $\mu(s)$ if t is $(\lambda\, s)$, $max(\mu(s_1), \mu(s_2))$ if t is $(s_1\ s_2)$, and $\mu(s) + \mu(e) + 1$ if t is $[\![s, ol, nl, e]\!]$.*

2. *For an environment e, $\mu(e)$ is 0 if e is nil, $max(\mu(s), \mu(e_1))$ if e is $(s, l) :: e_1$ and $\mu(e_1) + \mu(e_2) + 1$ if e is $\{\!\{e_1, nl, ol, e_2\}\!\}$.*

Definition 3.7.3. *The measures η_i on terms and environments for each natural number i are defined simultaneously by recursion as follows:*

1. *For a term t, $\eta_i(t)$ is 1 if t is a constant, a meta variable or a de Bruijn index, $\eta_i(s)+1$ if t is $(\lambda\, s)$, $max(\eta_i(s_1), \eta_i(s_2)) + 1$ if t is $(s_1\ s_2)$, and $\eta_{i+1}(s) + \eta_{i+1+\mu(s)}(e) + 1$ if t is $[\![s, ol, nl, e]\!]$.*

2. *For an environment e, $\eta_i(e)$ is 0 if e is nil, $max(\eta_i(s), \eta_i(e_1))$ if e is $(s, l) :: e_1$ and $\eta_{i+1}(e_1) + \eta_{i+1+\mu(e_1)}(e_2) + 1$ if e is $\{\!\{e_1, nl, ol, e_2\}\!\}$.*

Definition 3.7.4. *The translation $\mathcal{E}$ of suspension expressions to $\mathcal{T}$ is defined as follows:*

1. *For a term t, $\mathcal{E}(t)$ is $*$ if t is a constant a meta variable or a de Bruijn index, $app(\mathcal{E}(t_1), \mathcal{E}(t_2))$ if t is $(t_1\ t_2)$, $lam(\mathcal{E}(t'))$ if t is $(\lambda\, t')$ and $s_i(\mathcal{E}(t'), \mathcal{E}(e'))$ where $i = \eta_0(t)$ if t is $[\![t', ol, nl, e']\!]$.*

2. *For an environment e, $\mathcal{E}(e)$ is $*$ if e is nil, $cons(\mathcal{E}(t'), \mathcal{E}(e'))$ if e is $(t', l) :: e'$ and $s_i(\mathcal{E}(e_1), \mathcal{E}(e_2))$ where $i = \eta_0(e)$ if e is $\{\!\{e_1, nl, ol, e_2\}\!\}$.*

Using this translation we now lift our ordering on this collection of first-order terms to suspension expressions.

Definition 3.7.5 (Suspension Expression Order)**.** *For suspension expressions s and t we say $s \gg t$ if and only if $\mathcal{E}(s) \succ \mathcal{E}(t)$.*

There key properties of the $\succ$ relation carry over to the $\gg$ relation. First, subexpressions are smaller than their parent expressions. Second, the relation is monotonic in the sense that if v results from u by replacement of a subpart x by y such that $x \gg y$, then $u \gg v$. Third, the relation is wellfounded. These properties together with the following theorem will make this relation a powerful tool for performing induction over suspension expressions.

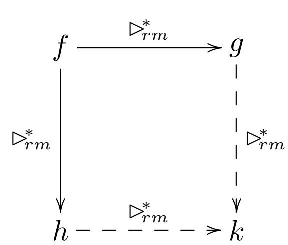
Theorem 3.7.1. *Every rewriting sequence based on the reading and merging rules terminates.*

Proof. A tedious but straightforward inspection of each of the reading and merging rules verifies the following: If $l \rightarrow r$ is an instance of these rules, then $l \gg r$, $\mu(l) \geq \mu(r)$, and, for every natural number i, $\eta_i(l) \geq \eta_i(r)$. Further, it is easily seen that if x and y are both either terms or environments such that $\mu(x) \geq \mu(y)$ and $\eta_i(x) \geq \eta_i(y)$ for each natural number i and if v is obtained from u by substituting y for x, then $\eta_i(u) \geq \eta_i(v)$ for each natural number i. From these observations it follows easily that if $t_1 \rhd_{rm} t_2$ then $t_1 \gg t_2$. The theorem is now a consequence of Lemma 3.7.1. $\square$

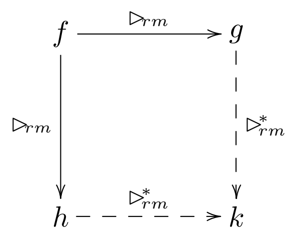
3.8 Confluence of Reading and Merging Rules

In this section we prove that the reading and merging rules are confluent, *i.e.*, that the choices we make in rewriting can always be reconciled. Thus $\rhd_{rm}$-normal forms are unique, which is another argument for the coherence of our reading and merging rules.

The property of confluence states that given terms f, g, and h such that $f \rhd^*_{rm} g$ and $f \rhd^*_{rm} h$, there exists a term k such that $g \rhd^*_{rm} k$ and $h \rhd^*_{rm} k$. This can be expressed using the diagrams described in Section 2.3.1 as,



A well known result (proved, for instance, in [Hue80]) in rewriting is that a terminating rewriting system is confluent if it is weakly confluent. Weak confluence is states that given terms f, g, and h such that $f \rhd_{rm} g$ and $f \rhd_{rm} h$, there exists a term k such that $g \rhd^*_{rm} k$ and $h \rhd^*_{rm} k$, *i.e.*, that the following figure holds.



This is much easier to show since we only need to consider one rewrite step from f to g and from f to h. In doing this, we must consider each possible overlap between any two rules. The most complicated of these is the overlap of (m1) with itself when applied to a term of the form

$$\llbracket\llbracket\llbracket t, ol_1, nl_1, e_1\rrbracket, ol_2, nl_2, e_2\rrbracket, ol_3, nl_3, e\rrbracket$$

This leads us to develop an associativity property for merged environments which is the content of the following section.

3.8.1 An Associativity Property for Environment Merging

Here we show that the following two environments rewrite to a common environment.

$$A = \{\!\{\{\!\{e_1, nl_1, ol_2, e_2\}\!\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}\!\}$$

$$B = \{\!\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\}\!\}$$

Essentially this tells us that to compute the effect of e_2 on e_1 and then compute the effect of e_3 on the result is the same as computing the effect of e_3 on e_2 and then computing the effect of that result on e_1. Ignoring the details for a moment, suppose that $e_1 = (t_1, n_1) :: e'_1$ and we are able to apply the rule (m6) to both terms (that is twice to A and once to B). Then the term portion of the environment term for A is roughly

$$\llbracket\llbracket t_1, ol_2, nl_2, e_2\rrbracket, ol_3, nl_3, e_3\rrbracket$$

and for B it is roughly

$$\llbracket t_1, ol_2 + (ol_3 \dot{-} nl_2), nl_3 + (nl_2 \dot{-} ol_3), \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\rrbracket$$

Then we can apply (m1) to bring these two terms back together. This is the heart of the proof. The vast majority of the proof is taken up by details and corner cases. For instance, we might not be able to apply (m6) to a term $[\![(t_1, n_1) :: e_1, nl_1, ol_2, e_2]\!]$ because $nl_1 > n_1$, e_2 is nil, or e_2 isn't of the form $(t_2, n_2) :: e_2'$. All of these cases must be handled and often doubly so since we have to apply (m6) twice to the term A. In order to ease these pains, we introduce a few lemmas which we can use to massage expressions into the proper form.

The rules (m4), (m5), and (m6) require their second environment to be of the form $(t, n) :: e$ which isn't the case with merged environments like in the term B. Ideally we could use the rules (m5) and (m6) to turn a term of the form $\{\!\{e, nl, ol, e'\}\!\}$ into one of the form $(t, n) :: e$, but if we do this we will not be able to use our inductive hypothesis. To accommodate this issue we introduce the following lemmas which essentially state that applying the rules (m5) and (m6) does not interfere with the process of rewriting.

Lemma 3.8.1. *Let A be the environment $\{\!\{e_1, nl_1, ol_1, \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\}\!\}$ where e_3 is a simple environment and e_2 is of the form $(t_2, n_2) :: e_2'$. Further, for any positive number i such that $i \leq nl_2 - n_2$ and $i \leq ol_3$, let B be the environment*

$$\{\!\{e_1, nl_1, ol_1, \{\!\{e_2, nl_2 - i, ol_3 - i, e_3\{i\}\}\!\}\}\!\}.$$

If $A \rhd_{rm}^* C$ for any simple environment C then also $B \rhd_{rm}^* C$.

Proof. It suffices to verify the claim when $i = 1$; an easy induction on i then extends the result to the cases where $i > 1$. For the case of $i = 1$, the argument is by induction on the length of the reduction sequence from A to C with the essential part being a consideration of the first rule used. The details are straightforward and hence omitted. □

Lemma 3.8.2. *Let A be the environment $\{\!\{e_1, nl_1, ol_1, \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\}\!\}$ where e_2 and e_3 are environments of the form $(t_2, nl_2) :: e_2'$ and $(t_3, n_3) :: e_3'$, respectively. Further, let B be the environment*

$$\{\!\{e_1, nl_1, ol_1, ([\![t_2, ol_3, n_3, e_3]\!], n_3 + (nl_2 \dot{-} ol_3)) :: \{\!\{e_2', nl_2, ol_3, e_3\}\!\}\}\!\}.$$

If $A \rhd_{rm}^* C$ for any simple environment C then also $B \rhd_{rm}^* C$.

Proof. The proof is again by induction on the length of the reduction sequence from A to C. The first rule in this sequence either produces B, in which case the lemma follows immediately, or it can be used on B (perhaps at more than one place) to produce a form that is amenable to the application of the induction hypothesis. □

In evaluating the composition of e_2 and e_3, it may be the case that some part of e_3 is inconsequential. The last observation that we need is that this part can be "pruned" immediately in calculating the composition of the combination of e_1 and e_2 with e_3. The following lemma is consequential in establishing this fact.

Lemma 3.8.3. *Let A be the environment $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ where e_2 is a simple environment.*

1. *If $ol_2 \leq nl_1 - lev(e_1)$ then A reduces to any simple environment that e_1 reduces to.*

2. *For any positive number i such that $i \leq nl_1 - lev(e_1)$ and $i \leq ol_2$, A reduces to any simple environment that $\{\!\{e_1, nl_1 - i, ol_2 - i, e_2\{i\}\}\!\}$ reduces to.*

Proof. Let e_1 be reducible to the simple environment e_1'. Then we may transform A to the form $\{\!\{e_1', nl_1, ol_2, e_2\}\!\}$. Recalling that the level of an environment is never increased by rewriting, we have that $lev(e_1') \leq lev(e_1)$. From this it follows that A can be rewritten to e_1' using rules (m5) and (m2) if $ol_2 \leq nl_1 - lev(e_1)$. This establishes the first part of the lemma.

The second part is nontrivial only if $nl_1 - lev(e_1)$ and ol_2 are both nonzero. Suppose this to be the case and let B be $\{\!\{e_1, nl_1 - 1, ol_2 - 1, e_2\{1\}\}\!\}$. The desired result follows by an induction on i if we can show that A can be rewritten to any simple environment that B reduces to. We do this by an induction on the length of the reduction sequence from B to the simple environment. This sequence must evidently be of length at least one. If a proper subpart of B is rewritten by the first rule in this sequence, then the same rule can be applied to A as well and the induction hypothesis easily yields the desired conclusion. If B is rewritten by one of the rules (m3)-(m6), then it must be the case that $A \rhd_{rm} B$ via either rule (m4) or (m5) from which the claim follows immediately. Finally, if B is rewritten using rule (m2), then $ol_2 \leq nl_1 - lev(e_1)$. The second part of the lemma is now a consequence of the first part. □

We now prove the associativity property for environment composition:

Lemma 3.8.4. *Let A and B be environments of the form*

$$\{\!\{\{\!\{e_1, nl_1, ol_2, e_2\}\!\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}\!\}$$

and

$$\{\!\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\}\!\},$$

respectively. Then there is a simple environment C such that $A \rhd_{rm}^* C$ and $B \rhd_{rm}^* C$.

Proof. We assume that e_1, e_2 and e_3 are simple environments; if this is not the case at the outset, then we may rewrite them to such a form in both A and B before commencing the proof we provide. Our argument is now based on an induction on the structure of e_3 with possibly further inductions on the structures of e_2 and e_1.

Base case for first induction. When e_3 is *nil*, the lemma is seen to be true by observing that both A and B rewrite to $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ by virtue of rule (m2).

Inductive step for first induction. Let $e_3 = (t_3, n_3) :: e_3'$. We now proceed by an induction on the structure of e_2.

Base case for second induction. When e_2 is *nil*, it can be seen that, by virtue of rules (m2), (m3) and either (m4) or (m5), A and B reduce to $\{\!\{e_1, nl_1, ol_3 - nl_2, e_3\{nl_2\}\}\!\}$ when $ol_3 > nl_2$ and to e_1 otherwise. The truth of the lemma follows immediately from this.

Inductive step for second induction. Let $e_2 = (t_2, n_2) :: e_2'$. We consider first the situation where $nl_1 > lev(e_1)$. Suppose further that $ol_3 \leq (nl_2 - n_2)$. Using rules (m5) and (m2), we see then that

$$B \rhd_{rm}^* \{\!\{e_1, nl_1, ol_2, e_2\}\!\}.$$

We also note that $ol_3 \leq (nl_2 + (nl_1 \dot{-} ol_2)) - lev(\{\!\{e_1, nl_1, ol_2, e_2\}\!\})$ in this case. Lemma 3.8.3 assures us now that A can be rewritten to any simple environment that $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ reduces to and thereby verifies the lemma in this case.

It is possible, of course, that $ol_3 > (nl_2 - n_2)$. Here we see that

$$B \rhd_{rm}^* \{\!\{e_1, nl_1 - 1, ol_2 + (ol_3 \dot{-} nl_2) - 1, \{\!\{e_2', n_2, ol_3 - (nl_2 - n_2), e_3\{nl_2 - n_2\}\}\!\}\}\!\}.$$

using rules (m5) and (m6). Using rule (m5), we also have that

$$A \rhd_{rm}^* \{\!\{\{\!\{e_1, nl_1 - 1, ol_2 - 1, e_2'\}\!\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}\!\}.$$

Invoking the induction hypothesis, it follows that A and

$$\{\!\{e_1, nl_1 - 1, ol_2 + (ol_3 \dot{-} nl_2) - 1, \{\!\{e_2', nl_2, ol_3, e_3\}\!\}\}\!\}$$

reduce to a common simple environment. By Lemma 3.8.1 it follows that B must also reduce to this environment.

The only remaining situation to consider, then, is that when $nl_1 = lev(e_1)$. For this case we need the last induction, that on the structure of e_1.

Base case for final induction. If e_1 is *nil*, then nl_1 must be 0. It follows easily that both A and B reduce to $\{\!\{e_2, nl_2, ol_3, e_3\}\!\}$ and that the lemma must therefore be true.

Inductive step for final induction. Here e_1 must be of the form $(t_1, nl_1) :: e_1'$. We dispense first with the situation where $n_2 < nl_2$. In this case, by rule (m5)

$$B \rhd_{rm}^{*} \{\!\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\!\{e_2, nl_2 - 1, ol_3 - 1, e_3'\}\!\}\}\!\}.$$

By the induction hypothesis used relative to e_3', B and the expression

$$\{\!\{\{\!\{e_1, nl_1, ol_2, e_2\}\!\}, nl_2 + (nl_1 \dot{-} ol_2) - 1, ol_3 - 1, e_3'\}\!\}$$

must reduce to a common simple environment. By Lemma 3.8.3, A must also reduce to this environment.

Thus, it only remains for us to consider the situation in which $n_2 = nl_2$. In this case by using rule (m1) twice we may transform A to the expression $A_h :: A_t$ where

$$A_h = ([\![[\![t_1, ol_2, n_2, e_2]\!], ol_3, n_3, e_3]\!], n_3 + ((nl_2 + (nl_1 \dot{-} ol_2)) \dot{-} ol_3))$$

and

$$A_t = \{\!\{\{\!\{e_1', nl_1, ol_2, e_2\}\!\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}\!\}.$$

Similarly, B may be rewritten to the expression $B_h :: B_t$ where

$$\begin{aligned} B_h = &([\![t_1, ol_2 + (ol_3 \dot{-} nl_2), n_3 + (nl_2 \dot{-} ol_3), \\ &\qquad ([\![t_2, ol_3, n_3, e_3]\!], n_3 + (nl_2 \dot{-} ol_3)) :: \{\!\{e_2', nl_2, ol_3, e_3\}\!\}]\!], \\ &\quad n_3 + (nl_2 \dot{-} ol_3) + (nl_1 \dot{-} (ol_2 + (ol_3 \dot{-} nl_2)))) \end{aligned}$$

and

$$B_t = \{\!\{e_1', nl_1, ol_2 + (ol_3 \dot{-} nl_2), ([\![t_2, ol_3, n_3, e_3]\!], n_3 + (nl_2 \dot{-} ol_3)) :: \{\!\{e_2', nl_2, ol_3, e_3\}\!\}\}\!\}.$$

Now, using straightforward arithmetic identities, it can be seen that the "index" components of A_h and B_h are equal. Further, the term component of A_h can be rewritten to a form identical to the term component of B_h by using the rules (m1) and (m6). Finally, by virtue of the induction hypothesis, it follows that A_t and the expression

$$\{\!\{e_1', nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\!\{e_2, nl_2, ol_3, e_3\}\!\}\}\!\}$$

reduce to a common simple environment. Lemma 3.8.2 allows us to conclude that B_t can also be rewritten to this expression. Putting all these observations together it is seen that A and B can be reduced to a common simple environment in this case as well. □

3.8.2 Proof of Confluence for Reading and Merging Rules

We are now in a position to prove the confluence of the reading and merging rules, using the ideas outlined in the beginning of this section.

Lemma 3.8.5. *The relation $\rhd_{rm}$ is weakly confluent.*

Proof. We recall the method of proof from [Hue80]. An expression t constitutes a nontrivial overlap of the rules R_1 and R_2 at a subexpression s if (a) t is an instance of the lefthand side of R_1, (b) s is an instance of the lefthand side of R_2 and also does not occur within the instantiation of a variable on the lefthand side of R_1 when this is matched with t and (c) either s is distinct from t or R_1 is distinct from R_2. Let r_1 be the expression that results from rewriting t using R_1 and let r_2 result from t by rewriting s using R_2. Then the pair $\langle r_1, r_2\rangle$ is called the conflict pair corresponding to the overlap in question. Relative to these notions, the theorem can be proved by establishing the following simpler property: for every conflict pair corresponding to the reading and merging rules, it is the case that the two terms can be rewritten to a common form using these rules.

In completing this line of argument, the nontrivial overlaps that we have to consider are those between (m1) and each of the rules (r1)-(r6), between (m1) and itself and between (m2) and (m3). The last of these cases is easily dealt with: the two expressions constituting the conflict pair are identical, both being *nil*. The overlap between (m1) and itself occurs over a term of the form $[\![[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!], ol_3, nl_3, e_3]\!]$. By using rule (m1) once more on each of the terms in the conflict pair, these can be rewritten to expressions of the form $[\![t, ol', nl', e']\!]$ and $[\![t, ol'', nl'', e'']\!]$, respectively, whence we can see that $ol' = ol''$ and $nl' = nl''$ by simple arithmetic reasoning and that e' and e'' reduce to a common form using Lemma 3.8.4. The overlaps between (m1) and the reading rules are also easily dealt with. For instance, consider the case of (m1) and (r2) where we have

$$[\![[\![\#i, 0, nl_1, nil]\!], ol_2, nl_2, e_2]\!]$$

Rewriting with (r2) first produces

$$[\![\#(i + nl_1), ol_2, nl_2, e_2]\!]$$

while rewriting with (m1) first yields

$$[\![\#i, ol_2 \dot{-} nl_1, nl_2 + (nl_1 \dot{-} ol_2), \{\!\{nil, nl_1, ol_2, e_2\}\!\}]\!].$$

For both expressions we can rewrite e_2 to a simple environment e_2' using Lemma 3.3.1. Now if $ol_2 \geq nl_1$ then both terms can be reconciled to

$$[\![\#i, ol_2 - nl_1, nl_2, e_2'\{nl_1\}]\!]$$

In the case of $ol_2 < nl_1$, both terms rewrite to $\#(i + nl_1 - ol_2)$. Thus the conflict pair is resolved. The other cases of overlaps between (m1) and the reading rules are similar and require roughly the same reasoning. □

As observed already, the main result of this section follows directly from Lemma 3.8.5 and Theorem 3.7.1.

Theorem 3.8.1. *The relation $\rhd_{rm}$ is confluent.*

The uniqueness of $\rhd_{rm}$-normal forms is an immediate consequence of Theorem 3.8.1. In the sequel, a notation for referring to such forms will be useful.

Definition 3.8.1 (Reading and merging normal form)**.** *The notation $|t|$ denotes the $\rhd_{rm}$-normal form of a suspension expression t.*

It is easily seen that the $\rhd_{rm}$-normal form for a term that does not contain meta variables is a a term that is devoid of suspensions, *i.e.*, a de Bruijn term. A further observation is that if the all the environments appearing in the original term are simple, then just the reading rules suffice in reducing it to the de Bruijn term that is its unique $\rhd_{rm}$-normal form.

3.9 Simulation of Beta Reduction

A fundamental property of all explicit substitution calculi is that they properly simulate the lambda calculus. In particular, we must ensure that our β_s rule corresponds to the β rule of the lambda calculus, modulo the reading and merging rules. The following two theorems establish this result. The first shows that a β_s rewrite on suspension terms corresponds to some number of β rewrites in the lambda calculus. One might think of this theorem as proving the soundness of our calculus. The second theorem shows that any β rewrite can be simulated using the rules of our calculus. One might think of this theorem as proving the completeness of our calculus.

Theorem 3.9.1. *If x_1 and x_2 are suspension expressions such that $x_1 \rhd_{\beta_s} x_2$ then $|x_1| \rhd_{\beta}^{*} |x_2|$.*

Proof. This result is proven for the original suspension calculus in [NW98]. We know from Theorem 3.4.1 that $\rhd_{rm}$-normal forms of x_1 and x_2 are the same as in the original suspension calculus, thus we can carry over the previous result. □

Theorem 3.9.2. *If x_1 and x_2 are suspension expressions in $\rhd_{rm}$-normal form such that $x_1 \rhd_\beta x_2$ then $x_1 \rhd^*_{rm\beta_s} x_2$.*

Proof. A stronger version of this property, where the result is replaced with $x_1 \rhd^*_{r\beta_s} x_2$, is proved as Lemma 8.2 of the original suspension paper. Since the β_s and the reading rules are essentially the same for the two calculi, the result carries over. □

3.10 Confluence of Overall Calculus

In this section we prove that the full suspension calculus is confluent even in the presence of graftable meta variables. One important distinction between the proof of this property and the proof of confluence for the reading and merging rules is that the latter is proven in spite of the merging rules while the former will be proven only because of the merging rules (see Section 3.6 for a discussion of why the merging rules are required). Thus this property speaks to the well designed nature of the merging rules. Moreover, this property makes the suspension calculus a candidate for unification procedures designed specifically for graftable meta variables [DHK95] because with this confluence property we are guaranteed that $\rhd_{rm\beta_s}$-normal forms are unique even with graftable meta variables.

For most explicit substitution calculi, confluence of the full calculus is proven using Hardin's Interpretation Method [Har89]. The interpretation method starts with the lambda calculus which is already confluent and it uses the confluence and strong termination of the substitution rules (in our case, the reading and merging rules) to close a confluence diagram for the overall calculus. This method is inadequate, however, when we allow for graftable meta variables. The problem is that the lambda calculus with graftable meta variables does not make sense and is not confluent. Instead, we follow the method presented in [CHL96] which is based the on the following key lemma.

Lemma 3.10.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be two relations defined on the same set X, $\mathcal{R}$ being confluent and strongly normalizing, and $\mathcal{S}$ being strongly confluent, i.e. such that the following diagrams hold for any $f, g, h \in X$:*

$$\frac{}{t \to t} \qquad \frac{}{e \to e}$$

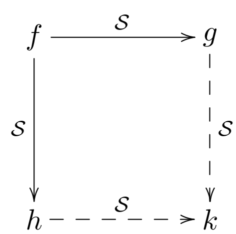
$$\frac{t_1 \to t_1' \qquad t_2 \to t_2'}{t_1\ t_2 \to t_1'\ t_2'} \qquad \frac{t \to t' \qquad e \to e'}{(t, l) :: e \to (t', l) :: e'}$$

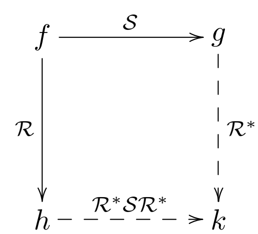
$$\frac{t \to t'}{\lambda\, t \to \lambda\, t'} \qquad \frac{e_1 \to e_1' \qquad e_2 \to e_2'}{\{\!\{e_1, nl_1, ol_2, e_2\}\!\} \to \{\!\{e_1', nl_1, ol_2, e_2'\}\!\}}$$

$$\frac{t \to t' \qquad e \to e'}{[\![t, ol, nl, e]\!] \to [\![t', ol, nl, e']\!]} \qquad \frac{t \to t'}{(t, n) \to (t', n)}$$

$$\frac{t_1 \to t_1' \qquad t_2 \to t_2'}{(\lambda\, t_1)\ t_2 \to [\![t_1', 1, 0, (t_2', 0) :: nil]\!]}$$

Figure 3.4: Parallel β_s-reduction





Then the relation $\mathcal{R}^*\mathcal{S}\mathcal{R}^*$ is confluent.

We will apply the lemma using the reading and merging rules as $\mathcal{R}$ and parallel β_s-reduction as $\mathcal{S}$.

Definition 3.10.1 (Parallel β_s-reduction)**.** *Parallel β_s-reduction is defined by the rules in Figure 3.4 and is denoted by $\rhd_{\beta_s||}$.*

Lemma 3.10.2. *$\rhd_{rm}$ and $\rhd_{\beta_s||}$ satisfy the condition of Lemma 3.10.1.*

Proof. $\rhd_{\beta_s||}$ is obviously strongly confluent since $\rhd_{\beta_s}$ is is a left linear system with no critical pairs. This proves that the first diagram in Lemma 3.10.1 holds.

For the second diagram, the interesting case is the critical pair for $f = [\![(\lambda\, t_1)\ t_2, ol, nl, e]\!]$. In this case, we have $g = [\![[\![t_1', 1, 0, (t_2', 0) :: nil]\!], ol, nl, e']\!]$ and $h = [\![\lambda\, t_1, ol, nl, e]\!]\ [\![t_2, ol, nl, e]\!]$, where $t_1 \rhd_{\beta_s||} t_1'$, $t_2 \rhd_{\beta_s||} t_2'$, and $e \rhd_{\beta_s||} e'$. We must find a k such that $g \rhd_{rm}^* k$ and $h \rhd_{rm}^* h' \rhd_{\beta_s||} h'' \rhd_{rm}^* k$. This is straightforward since

$$\begin{aligned} g &= [\![[\![t_1', 1, 0, (t_2', 0) :: nil]\!], ol, nl, e']\!] \\ &\rhd_m [\![t_1', ol+1, nl, \{\!\{ (t_2', 0) :: nil, 0, ol, e' \}\!\}]\!] \\ &\rhd_{rm}^* [\![t_1', ol+1, nl, ([\![t_2', ol, nl, e']\!], nl) :: e']\!] \end{aligned}$$

and

$$\begin{aligned} h &= [\![\lambda\, t_1, ol, nl, e]\!]\ [\![t_2, ol, nl, e]\!] \\ &\rhd_r \lambda\, [\![t_1, ol+1, nl+1, (\#1, nl+1) :: e]\!]\ [\![t_2, ol, nl, e]\!] \\ &\rhd_{\beta_s||} [\![[\![t_1', ol+1, nl+1, (\#1, nl+1) :: e']\!], 1, 0, ([\![t_2', ol, nl, e']\!], 0) :: nil]\!] \\ &\rhd_m [\![t_1', ol+1, nl, \{\!\{ (\#1, nl+1) :: e', nl+1, 1, ([\![t_2', ol, nl, e']\!], 0) :: nil \}\!\}]\!] \\ &\rhd_{rm}^* [\![t_1', ol+1, nl, ([\![t_2', ol, nl, e']\!], nl) :: e']\!] \end{aligned}$$

□

Theorem 3.10.1. *The relation $\rhd_{rm\beta_s}$ is confluent.*

Proof. Note that $\rhd_{rm\beta_s} \subseteq \mathcal{R}^* \mathcal{S} \mathcal{R}^* \subseteq \rhd_{rm\beta_s}^*$. □

3.11 Similarity in the Suspension Calculus

The purpose of this section is to introduce a notion of similarity in the suspension calculus which relates suspension expressions that differ only in the renumbering and indices of environment terms. This allows us to formally capture the notion that two environments are similar enough that they act the same during rewriting which will be useful when we translate from another explicit substitution calculus into the suspension calculus (Section 4.2.2). The notion of similarity stems from the fact that there are two ways to represent the renumbering to be done on an environment term. One is using the difference between the new embedding level of the suspension and the embedding level of the environment term. The other is with an explicit renumbering substitution applied to the term in the environment term. This section proves that these two notions are equivalent for the purpose of finding normal forms.

Definition 3.11.1. *The similarity relation $\sim$ is defined in Figure 3.5.*

$$\frac{}{t \sim t} \qquad \frac{}{e \sim e}$$

$$\frac{t_1 \sim t_1' \qquad t_2 \sim t_2'}{t_1\ t_2 \sim t_1'\ t_2'} \qquad \frac{t \sim t' \qquad e \sim e'}{(t, n) :: e \sim (t', n) :: e'}$$

$$\frac{t \sim t'}{\lambda\, t \sim \lambda\, t'} \qquad \frac{e_1 \sim e_1' \qquad e_2 \sim e_2'}{\{\!\!\{e_1, nl_1, ol_2, e_2\}\!\!\} \sim \{\!\!\{e_1', nl_1, ol_2, e_2'\}\!\!\}}$$

$$\frac{t \sim t' \qquad e \sim e'}{[\![t, ol, nl, e]\!] \sim [\![t', ol, nl, e']\!]} \qquad \frac{t \sim t'}{(t, n) \sim (t', n)}$$

$$\frac{t \sim t' \qquad r \sim r' \qquad e \sim e'}{([\![t, ol, nl, r]\!], nl + k) :: e \sim ([\![t', ol, nl', r']\!], nl' + k) :: e'}$$

Figure 3.5: The similarity relation, $\sim$

The main result of this section is to prove that similar terms rewrite to the same normal form. This first requires proving the following lemma.

Lemma 3.11.1. *Let* $\{\!\!\{e_1, nl_1, ol_2, e_2\}\!\!\} \rhd_{rm}^* e$ *where* e *is a simple environment.*

- *If* $nl_1 - lev(e_1) \geq k$ *and* $ol_2 \geq k$*, then* $\{\!\!\{e_1, nl_1 - k, ol_2 - k, e_2\{k\}\}\!\!\} \rhd_{rm}^* e$.
- *If* $nl_1 - lev(e_1) \geq ol_2$*, then* $e_1 \rhd_{rm}^* e$.

Proof. The proof is by induction on the length of the sequence $\{\!\!\{e_1, nl_1, ol_2, e_2\}\!\!\} \rhd_{rm}^* e$. □

Theorem 3.11.1. *If* $t \sim t'$ *for terms* t *and* t' *then they rewrite by reading and merging rules to the same de Bruijn term. If* $e \sim e'$ *for environments* e *and* e' *then they rewrite by reading and merging rules to similar simple environments.*

Proof. We prove the general case of $exp \sim exp'$ for suspension expressions exp and exp'. We do this by induction using the relation $\gg$ defined in Definition 3.7.5. Note that this relation decreases when an expression is rewritten using the reading and merging rules and also a subpart is always smaller than the original expression.

Because we are inducting using $\gg$, we can assume that the result already holds for any similar subparts of exp and exp'. Then we can rewrite these similar subparts to be equal in the case of terms or simple environments in the case of environments. This decreases the measure of the overall terms exp and exp' and thus the inductive hypothesis then applies to them. By this reasoning, we can assume that whenever two subparts of exp and exp' are similar they are in fact equal in the case of terms and simple in the case of environments. We will also use the convention that x and x' are always similar.

Let us consider cases based on the structure of exp and exp', first looking at the case when both are terms. If they are both constants or de Bruijn indices then the result is trivial. If $exp = (t_1\ t_2)$ then $exp' = (t'_1\ t'_2)$ and $t_1 = t'_1$ and $t_2 = t'_2$ so the result follows trivially. A similar result holds in the case where exp and exp' are lambda abstractions.

The first nontrivial case is when exp and exp' are both suspensions, say $exp = [\![t, ol, nl, e]\!]$ and $exp' = [\![t, ol, nl, e']\!]$. Now consider which rewrite rules apply to the toplevel of these terms, keeping in mind that t is in normal form. If t is an application or an abstraction then (r5) or (r6) applies and the result follows from the inductive hypothesis. If t is a de Bruijn term and (r2) or (r4) applies then the result again follows from the inductive hypothesis. If (r3) applies and e and e' have the same head then the result is trivial. The key case is when (r3) applies and e and e' have different heads, in which case we have,

$$\begin{aligned} exp &= [\![\#1, ol, nl, ([\![t_r, ol_r, nl_r, r]\!], nl_r + k) :: e_1]\!] \\ &\rhd_{(r3)} [\![[\![t_r, ol_r, nl_r, r]\!], 0, nl - (nl_r + k), nil]\!] \\ &\rhd_{(m1)} [\![t_r, ol_r, nl - (nl_r + k) + nl_r, \{\!\{r, nl_r, 0, nil\}\!\}]\!] \\ &\rhd_{(m2)} [\![t_r, ol_r, nl - k, r]\!] \end{aligned}$$

$$\begin{aligned} exp' &= [\![\#1, ol, nl, ([\![t_r, ol_r, nl'_r, r']\!], nl'_r + k) :: e'_1]\!] \\ &\rhd_{(r3)} [\![[\![t_r, ol_r, nl'_r, r']\!], 0, nl - (nl'_r + k), nil]\!] \\ &\rhd_{(m1)} [\![t_r, ol_r, nl - (nl'_r + k) + nl'_r, \{\!\{r', nl'_r, 0, nil\}\!\}]\!] \\ &\rhd_{(m2)} [\![t_r, ol_r, nl - k, r']\!] \end{aligned}$$

The two resulting suspensions are similar and smaller than their original terms, thus the inductive hypothesis finishes this case.

The other half of the proof is to show that when exp and exp' are similar environments then they rewrite to similar simple environments. The cases when exp and exp' are either nil or a cons follow trivially from the inductive hypothesis. The important case is when $exp = \{\!\{e_1, nl_1, ol_2, e_2\}\!\}$ and $exp' = \{\!\{e'_1, nl_1, ol_2, e'_2\}\!\}$. Consider the cases for which rewrites can apply to the toplevel of both expressions. If (m2), (m3), or (m4) applies to the first expression then the same rewrite applies to the second environment and the result follows easily. The case when (m5) applies to both is also direct using the inductive hypothesis. The two remaining cases are the most interesting: when (m6) applies to both, and when (m5) applies to one and (m6) to the other.

Consider when (m6) applies to both expressions. Here the head terms of e_1 and e_1' must be the same. If the head terms of e_2 and e_2' are also the same then the result is trivial. Otherwise we have,

$$\begin{aligned} exp &= \{\!\{(t_1, nl_1) :: e_3, nl_1, ol_2, ([\![t_r, ol_r, nl_r, r]\!], nl_r + k) :: e_4\}\!\} \\ &\rhd_{(m6)} ([\![t_1, ol_2, nl_r + k, ([\![t_r, ol_r, nl_r, r]\!], nl_r + k) :: e_4]\!], nl_r + k + (nl_1 \dot{-} ol_2)) \\ &\qquad :: \{\!\{e_3, nl_1, ol_2, ([\![t_r, ol_r, nl_r, r]\!], nl_r + k) :: e_4\}\!\} \end{aligned}$$

$$\begin{aligned} exp' &= \{\!\{(t_1, nl_1) :: e_3', nl_1, ol_2, ([\![t_r', ol_r, nl_r', r']\!], nl_r' + k) :: e_4'\}\!\} \\ &\rhd_{(m6)} ([\![t_1, ol_2, nl_r' + k, ([\![t_r', ol_r, nl_r', r']\!], nl_r' + k) :: e_4']\!], nl_r' + k + (nl_1 \dot{-} ol_2)) \\ &\qquad :: \{\!\{e_3', nl_1, ol_2, ([\![t_r, ol_r, nl_r', r']\!], nl_r' + k) :: e_4'\}\!\} \end{aligned}$$

These two environments are still similar and the inductive hypothesis now applies.

The final case is when (m5) applies to one expression and (m6) to the other. Without loss of generality, assume that (m6) applies to exp and (m5) to exp'. There are two subcases based on whether the heads of e_2 and e_2' are the same or not. Here we will only consider hardest case where the heads differ. The other case is a simplification of the following argument,

$$\begin{aligned} exp &= \{\!\{([\![t_s, ol_s, nl_s, s]\!], nl_s + k_s) :: e_3, nl_1, ol_2, ([\![t_r, ol_r, nl_r, r]\!], nl_r + k_r) :: e_4\}\!\} \\ &\rhd_{(m6)} ([\![[\![t_s, ol_s, nl_s, s]\!], ol_2, nl_r + k_r, e_2]\!], nl_r + k_r + (nl_1 \dot{-} ol_2)) :: \{\!\{e_3, nl_1, ol_2, e_2\}\!\} \\ &\rhd_{(m1)} ([\![t_s, ol_s + (ol_2 \dot{-} nl_s), nl_r + k_r + (nl_s \dot{-} ol_2), \{\!\{s, nl_s, ol_2, e_2\}\!\}]\!], \\ &\qquad nl_r + k_r + (nl_1 \dot{-} ol_2)) :: \{\!\{e_3, nl_1, ol_2, e_2\}\!\} \end{aligned}$$

$$exp' = \{\!\{([\![t_s', ol_s, nl_s', s']\!], nl_s' + k_s) :: e_3', nl_1, ol_2, e_2'\}\!\}$$

Consider first the case when $nl_1 - (nl_s' + k_s) \geq ol_2$. Since (m6) applies to the first expression we know that $nl_1 = nl_s + k_s$ and thus $nl_s - nl'_s \geq ol_2$. The first expression then rewrites to

$$([\![t_s, ol_s, nl_r + k_r + nl_s - ol_2, \{\!\{s, nl_s, ol_2, e_2\}\!\}]\!], \\ nl_r + k_r + nl_1 - ol_2)) :: \{\!\{e_3, nl_1, ol_2, e_2\}\!\}$$

The second expression rewrites using (m5) multiple times to

$$([\![t'_s, ol_s, nl'_s, s']\!], nl'_s + k_s) :: e'_3$$

It is easily seen that $exp \gg \{\!\{s, nl_s, ol_2, e_2\}\!\}$ and $exp' \gg \{\!\{s', nl_s, ol_2, e'_2\}\!\}$. Moreover, $\{\!\{s, nl_s, ol_2, e_2\}\!\}$ and $\{\!\{s', nl_s, ol_2, e'_2\}\!\}$ are similar so the inductive hypothesis applies and tells us these merged environments rewrite to similar simple environments. Since $nl_s - ol_2 \geq nl'_s \geq lev(s')$, applying Lemma 3.11.1 yields that $\{\!\{s, nl_s, ol_2, e_2\}\!\}$ and s' also rewrite to similar simple environments. By applying these rewrites, the heads of our two environments are now similar. By the inductive hypothesis we also know that $\{\!\{e_3, nl_1, ol_2, e_2\}\!\}$ and $\{\!\{e'_3, nl_1, ol_2, e'_2\}\!\}$ rewrite to similar simple environments. Since $nl_1 - lev(e'_3) \geq ol_2$ we can apply Lemma 3.11.1 to know that $\{\!\{e_3, nl_1, ol_2, e_2\}\!\}$ and e'_3 rewrite to similar simple environments, thus finishing this case.

The other case is when $nl_1 - (nl'_s + k_s) < ol_2$. Then we can apply (m5) multiple times to exp' and then eventually (m6),

$$\begin{aligned} exp' &= \{\!\{([\![t'_s, ol_s, nl'_s, s']\!], nl'_s + k_s) :: e'_3, nl_1, ol_2, e'_2\}\!\} \\ &\rhd^*_{(m5)} \{\!\{([\![t'_s, ol_s, nl'_s, s']\!], nl'_s + k_s) :: e'_3, nl'_s + k_s, ol_2 - nl_s + nl'_s, \\ &\qquad e'_2\{nl_s - nl'_s\}\}\!\} \\ &\rhd_{(m6)} ([\![[\![t'_s, ol_s, nl'_s, s']\!], ol_2 - nl_s + nl'_s, l, e'_2\{nl_s - nl'_s\}]\!], \\ &\qquad l + (k_s \dot{-} (ol_2 - nl_s))) :: \\ &\qquad\qquad \{\!\{e'_3, nl'_s + k_s, ol_2 - nl_s + nl'_s, e'_2\{nl_s - nl'_s\}\}\!\} \end{aligned}$$

Where $l = lev(e'_2\{nl_s - nl'_s\})$. Let us focus first on the tail portions of our two environments. By the inductive hypothesis $\{\!\{e_3, nl_1, ol_2, e_2\}\!\}$ and $\{\!\{e'_3, nl_1, ol_2, e'_2\}\!\}$ rewrite to similar simple environments. Then by applying Lemma 3.11.1, $\{\!\{e_3, nl_1, ol_2, e_2\}\!\}$ and $\{\!\{e'_3, nl'_s + k_s, ol_2 - nl_s + nl'_s, e'_2\{nl_1 - nl'_s\}\}\!\}$ rewrite to similar simple environments.

Finally we focus on the heads of our environments. The head for exp' can now be rewritten using (m1),

$$\begin{aligned}&[\![[\![t'_s, ol_s, nl'_s, s']\!], ol_2 - nl_s + nl'_s, l, e'_2\{nl_s - nl'_s\}]\!]\\ &\rhd_{(m1)} [\![t'_s, ol_s + ol_2 - nl_s, l, \{\!\{s', nl'_s, ol_2 - nl_s + nl'_s, e'_2\{nl_s - nl'_s\}\}\!\}]\!]\end{aligned}$$

Note that as before $exp \gg \{\!\{s, nl_s, ol_2, e_2\}\!\}$ and $exp' \gg \{\!\{s', nl_s, ol_2, e'_2\}\!\}$, also $\{\!\{s, nl_s, ol_2, e_2\}\!\}$ and $\{\!\{s', nl_s, ol_2, e'_2\}\!\}$ are similar, so by the inductive hypothesis these merged environments rewrite to similar simple environments. Since $nl_s - lev(s') \geq nl_s - nl'_s$ and $ol_2 \geq nl_s - nl'_s$, applying Lemma 3.11.1 yields that $\{\!\{s, nl_s, ol_2, e_2\}\!\}$ and $\{\!\{s', nl'_s, ol_2 - nl_s + nl'_s, e'_2\{nl_s - nl'_s\}\}\!\}$ rewrite to similar simple environments. Using this rewriting results in the heads being similar. □

Chapter 4

Comparison with Other Explicit Substitution Calculi

There are many explicit substitution calculi that offer alternative treatments of substitutions which leads to varying computational properties. In this chapter we outline three computational properties that are of particular interest and use these to categorize various calculi based on how well they capture these. The three properties we focus on are (1) combination of substitution walks, (2) confluence in the presence of graftable meta variables, and (3) preservation of strong normalization.

Combination of substitution walks, also called merging, can be traced back to de Bruijn [dB72]. The substitution operation on de Bruijn terms, see Definition 2.2.2, is denoted by $S(t; s_1, s_2, \ldots)$ and represents the term t where s_i is substituted for the the i^{th} de Bruijn index. De Bruijn establishes the meta-property $S(S(t; s_1, s_2, \ldots); r_1, r_2, \ldots) = S(t, u_1, u_2, \ldots)$ where $u_i = S(s_i, r_1, r_2, \ldots)$. Here two substitutions walks over t are merged into a single substitution walk over t. For a more concrete example, consider the term $((\lambda\,\lambda\, t_1)\ t_2\ t_3)$. A naive reduction of this term would require two walks over the structure of t_1: the first for t_2 and the second for t_3. Moreover, the second walk would also have to walk over the structure of t_2 each place where it is substituted into t_1. A more reasonable approach is to merge the two substitution prior to making a walk over the structure of t_1. For instance, in the suspension calculus we can rewrite the term to $[\![t_1, 2, 0, (t_2, 0) :: (t_3, 0) :: nil]\!]$ which requires only one walk over t_1 and avoids any walks over t_2 since the non-overlapping nature of the two substitutions is detected by the merging process. In practice, this property has proven to have a great impact on efficiency [LNQ04].

Confluence in the presence of graftable meta variables, see Section 3.6, requires a calculus with rules interactions between substitutions. To see why, consider the example from Section 3.6 in a named context where we have the term $((\lambda a.((\lambda b.X)\ t_1))\ t_2)$ with X a graftable meta variable. Depending on which redex is contracted first, this term can reduce to either $X\langle t_1/b\rangle\langle t_2/a\rangle$ or $X\langle t_2/a\rangle\langle t_1\langle t_2/a\rangle/b\rangle$ where $\langle t/x\rangle$ is an explicit representation of substitution. In order to reconcile these two terms, interaction rules for substitutions must be added to the calculus. These interaction rules either take the form of combination rules, as seen in the previous paragraph, or permutation rules.

Preservation of strong normalization (PSN) means that lambda terms which are strongly normalizing in the lambda calculus remain strongly normalizing in the explicit substitution calculus, *i.e.*, an infinite reduction path is never added to a term with only finite reduction paths. To see why PSN might fail, consider the above problem of confluence and how we might resolve it by adding a permutation rule of the form $X\langle t_1/b\rangle\langle t_2/a\rangle \rightarrow X\langle t_2/a\rangle\langle t_1\langle t_2/a\rangle/b\rangle$. This rule fixes the confluence problem, but now the system fails to preserve strong normalization since the new rule can be repeatedly applied to itself to permute the substitutions back and forth. Preservation of strong normalization is desirable because one often works in a representational setting with typed lambda terms which are strongly normalizing and if PSN holds then all of those terms remain strongly normalizing in the explicit substitution calculus. On the other hand, if PSN does not hold then one must be careful in selecting a reduction strategy which avoids the newly introduced infinite reduction paths. Preservation of strong normalization is studied in further depth in Chapter 5.

Another part of our survey of explicit substitution calculi consists of translations between the other popular calculi and the suspension calculus towards understanding and contrasting their relative capabilities. To give substance to our translations, we established relevant properties of the translations such as their correctness and their ability to preserve important computational properties of the calculi they relate. The first half of showing correctness is that well-formed terms are translated to well-formed terms. The second half is that normal forms, with respect to substitutions, are preserved by the translation. To show that important properties are preserved, we will argue that the translations are information preserving, which is an intuitive, rather than formal notion. There are various ways in which we can capture this notion with the most desirable being to show that if t rewrites to r in one step then given the translation T, $T(t)$ rewrites to $T(r)$ in at least one step. We call this property *simulation* because it shows the translation preserves the information needed to simulate the substitution process of one calculus using the substitution process of another. This is not always possible due to the idiosyncrasies of different calculi. In these cases we will look we will find other ways of arguing for information preservation, while also looking at why simulation fails since the reason often reveals key differences between calculi. Finally, we note that for all of our translations we assume we are in a context without graftable meta variables.

We begin by separating the calculi based on combination of substitution walks, since this property is evident and has the greatest effect on the syntax of the language.

4.1 Calculi Without Merging

In this section we look at three calculi without merging: $\lambda\upsilon$-calculus [BBLRD96], λs-calculus [KR95], and λs_e-calculus [KR97]. These calculi lack the syntax for merging substitutions and instead each substitution in these calculi represents a substitution for at most one de Bruijn index and then possible renumberings for other de Bruijn indices. Since the notion of substitution in the suspension calculus is more general, we can only discuss a translation from these calculi to the suspension calculus and not the other way around. Nevertheless, seeing how the substitution concepts in these calculi are reflected in the suspension calculus gives us greater insight into their key characteristics.

4.1.1 The $\lambda\upsilon$-calculus

The $\lambda\upsilon$-calculus is actually a simplification of the $\lambda\sigma$-calculus, a calculus we will see more of in Section 4.2. The $\lambda\upsilon$-calculus was created by removing the syntax for merging of substitutions available in the $\lambda\sigma$-calculus and then modifying the rewriting rules to accommodate the new syntax. This simplified system was proven to preserve strong normalization, but at the cost of confluence in the presence of graftable meta variables. Another cost of the simplification is a peculiarity in the rewriting rules that make the system undesirable from an implementation perspective. We develop these issues in this section, starting first with the syntax of the calculus.

Definition 4.1.1. *The syntax of $\lambda\upsilon$-expressions is given by the following definitions of terms, denoted a and b, and substitutions, denoted s.*

$$\begin{array}{lcl} a & ::= & \underline{n} \mid a\ b \mid \lambda\, a \mid a[s] \\ s & ::= & a/ \mid \Uparrow\!(s) \mid \ \uparrow \end{array}$$

The term $\underline{n}$ represents the n^{th} de Bruijn index, and $a[s]$ is called a *closure*. The substitution $a/$ is called *slash* and represents the substitution of a for the first de Bruijn index and a shifting down of all other de Bruijn indices. The substitution $\Uparrow\!(s)$ is called *lift* and is used to push substitutions underneath lambda abstractions. The last substitution $\uparrow$ is called *shift* and represents increasing all free de Bruijn indices by one. Many of these concepts are less generalized versions of what is available in the suspension calculus, a point we make more explicit with the following translation.

Definition 4.1.2. *The translation T from $\lambda\upsilon$-terms to suspension terms and the translation E from $\lambda\upsilon$-substitutions to triples of an old embedding level, a new embedding level, and a suspension environment are defined simultaneously by recursion as follows:*

(B)	$(\lambda\, a)\ b \to a[b/]$	(VarShift)	$\underline{n}[\uparrow] \to \underline{n+1}$
(App)	$(a\ b)[s] \to a[s]\ b[s]$	(FVarLift)	$\underline{1}[\Uparrow(s)] \to \underline{1}$
(Lambda)	$(\lambda\, a)[s] \to \lambda\, a[\Uparrow(s)]$	(RVarLift)	$\underline{n+1}[\Uparrow(s)] \to \underline{n}[s][\uparrow]$
(FVar)	$\underline{1}[a/] \to a$		
(RVar)	$\underline{n+1}[a/] \to \underline{n}$		

Figure 4.1: Rewrite rules for the $\lambda\upsilon$-calculus

1. *For a term t, $T(t)$ is $\#n$ if t is $\underline{n}$, $(T(a)\ T(b))$ if t is $(a\ b)$, $\lambda\, T(a)$ if t is $\lambda\, a$, and $[\![T(a), ol, nl, e]\!]$ if t is $a[s]$ where $(ol, nl, e) = E(s)$.*

2. *For a substitution s, $E(s)$ is $(1, 0, (T(a), 0) :: nil)$ if s is $a/$, $(0, 1, nil)$ if s is $\uparrow$, and $(ol + 1, nl + 1, (\#1, nl + 1) :: e)$ if s is $\Uparrow(s')$ where $(ol, nl, e) = E(s')$.*

Theorem 4.1.1. *For every $\lambda\upsilon$-term a, $T(a)$ is a well-formed suspension term.*

Proof. The proof is by induction using the dual property that for every $\lambda\upsilon$-substitution s such that $(ol, nl, e) = E(s)$ we have $ol = len(e)$, $nl \geq lev(e)$, and e is a well-formed suspension environment. □

The rules of the $\lambda\upsilon$-calculus are presented in Figure 4.1. We define the υ rules to be all the rules of the $\lambda\upsilon$-calculus except (B). Because there is no possibility for merging substitutions, the υ rules simply push substitutions down in the tree and then evaluate them once they are applied to de Bruijn indices. Thus most of the υ rules can be matched up with a corresponding reading rule from the suspension calculus, with the exception being the rule RVarLift. The fundamental problem with this rule is that it replaces a single substitution on the left with two substitutions on the right. From the suspension calculus point of view, this is a step backwards. Thus we instead prove the following theorem in which $a \rhd_\upsilon b$ implies $T(a)$ and $T(b)$ rewrite to a common term rather than a stronger one in which $T(a) \rhd^*_{rm} T(b)$.

Theorem 4.1.2. *Let a and b be $\lambda\upsilon$-terms such that $a \rhd_\upsilon b$. Then there exists a suspension term t such that $T(a) \rhd^*_{rm} t$ and $T(b) \rhd^*_{rm} t$.*

Proof. The proof is by case analysis on the rule used to transition from a to b. In every case but RVarLift we can actually prove that $T(a) \rhd^*_{rm} T(b)$. The most difficult of these cases is FVar for which we must show $[\![\#1, 1, 0, (T(a), 0) :: nil]\!] \rhd^*_{rm} T(a)$. To do this, we first apply (r3) to generate $[\![T(a), 0, 0, nil]\!]$. Then we prove by induction the general property that $[\![t, 0, 0, nil]\!] \triangleright^*_{rm} t$ in a setting without graftable meta variables.

In the case of RVarLift, suppose that $(ol, nl, e) = E(s)$. Then we can show that the terms $[\![\#(n+1), ol+1, nl+1, (\#1, nl+1) :: e]\!]$ and $[\![[\![\#n, ol, nl, e]\!], 0, 1, nil]\!]$ have a common reduct in the term $[\![\#n, ol, nl+1, e]\!]$. □

Based on this theorem, the translation T preserves de Bruijn normal forms. To show that T is information preserving we offer the following theorem which shows that T is one-to-one.

Theorem 4.1.3. *The translation T is one-to-one.*

Proof. The proof is by induction using the dual property that E is one-to-one. □

Looking again at the RVarLift rule, we can see a problem from the implementation perspective. Consider the term $\underline{4}[\Uparrow(\Uparrow(\Uparrow(a/)))]$ which rewrites to $a[\uparrow][\uparrow][\uparrow]$. Here three separate renumbering passes are generated in order to increase all free de Bruijn indices by three. The problem is that not only is combination of substitutions not allowed, but the syntax of the $\lambda\upsilon$-calculus is not rich enough to encode a renumbering of de Bruijn indices by anything but one. In the next section we will see another calculus without merging, but with a more general notion of substitution which avoids this problem.

4.1.2 The λs-calculus

The λs-calculus is similar to the $\lambda\upsilon$-calculus in that it preserves strong normalization and fails to have confluence in the presence of graftable meta variables. The two primary differences are that the λs-calculus clearly separates the processes of substitution and renumbering, and the λs-calculus has more general notion of substitution. These differences are reflected in the syntax.

Definition 4.1.3. *The syntax of λs-expressions is given by the following definition of terms, denoted a and b.*

$$a \quad ::= \quad \mathtt{n} \mid a\ b \mid \lambda a \mid a\,\sigma^i\,b \mid \varphi^i_k\,a$$

Here $\mathtt{n}$ and i range over all positive integers and k over all non-negative integers.

The term $\mathtt{n}$ represents the n^{th} de Bruijn index. The term $a\,\sigma^i\,b$ is called a *closure* and represents the substitution of a renumbered version of b for the i^{th} de Bruijn index in a and

σ-generation	$(\lambda\, a)\ b \rightarrow a\, \sigma^1\, b$
σ-λ-transition	$(\lambda\, a)\, \sigma^i\, b \rightarrow \lambda\, (a\, \sigma^{i+1}\, b)$
σ-app-transition	$(a_1\ a_2)\, \sigma^i\, b \rightarrow (a_1\, \sigma^i\, b)\ (a_2\, \sigma^i\, b)$
σ-destruction	$\mathtt{n}\, \sigma^i\, b \rightarrow \begin{cases} \mathtt{n-1} & \text{if } n > i \\ \varphi^i_0\, b & \text{if } n = i \\ \mathtt{n} & \text{if } n < i \end{cases}$
φ-λ-transition	$\varphi^i_k(\lambda\, a) \rightarrow \lambda\, (\varphi^i_{k+1}\, a)$
φ-app-transition	$\varphi^i_k(a_1\ a_2) \rightarrow (\varphi^i_k\, a_1)\ (\varphi^i_k\, a_2)$
φ-destruction	$\varphi^i_k\, \mathtt{n} \rightarrow \begin{cases} \mathtt{n+i-1} & \text{if } n > k \\ \mathtt{n} & \text{if } n \leq k \end{cases}$

Figure 4.2: Rewrite rules for the λs-calculus

a shifting down by one of all de Bruijn indices greater than i in a. The term $\varphi^i_k\, a$ is called an *update* and represents an increase by $i-1$ of all de Bruijn indices greater than k. All of these concepts can be translated into the suspension calculus by the following translation.

Definition 4.1.4. *The translation T from λs-terms to suspension terms is defined by recursion as follows: For a term t, $T(t)$ is $\#n$ if t is $\mathtt{n}$, $(T(a)\ T(b))$ if t is $(a\ b)$, $\lambda\, T(a)$ if t is $\lambda\, a$, $[\![T(a), i, i-1, (\#1, i-1) :: (\#1, i-2) :: \ldots :: (\#1, 1) :: (T(b), 0) :: nil]\!]$ if t is $a\, \sigma^i\, b$, and $[\![T(a), k, k+i-1, (\#1, k+i-1) :: (\#1, k+i-2) :: \ldots :: (\#1, i) :: nil]\!]$ if t is $\varphi^i_k\, a$.*

Theorem 4.1.4. *For every λs-term a, $T(a)$ is a well-formed suspension term.*

Proof. The proof is by induction. □

The rules of the λs-calculus are presented in Figure 4.2. We define the s rules to be all the rules of the λs-calculus except *σ-generation*. Because of the separation between substitution and renumbering, there is some redundancy in the rules, *e.g.* *σ-app-transition* and *φ-app-transition*. But looking at *σ-λ-transition* and *φ-λ-transition*, we see the benefit is that substitution and renumbering can have separate behaviors for descending underneath lambda abstractions. This cleanness in the rules allows for very well behaved translation.

Theorem 4.1.5. *Let a and b be λs-terms such that $a \rhd_s b$. Then $T(a) \rhd^+_r T(b)$.*

Proof. The proof is by case analysis on the rule used to transition from a to b. □

The above theorem tells us that the translation T is information and normal form preserving. Moreover, it shows us that the suspension calculus, even without merging, is capable of exactly simulating the λs-calculus, and it gives us a proof that the s rules of the λs-calculus are strongly normalizing since the reading (and merging) rules are strongly normalizing. In the original paper on the λs-calculus a similar translation is proven from the λs-calculus to the $\lambda\sigma$-calculus, and it is through this translation that the strong normalization of the s rules is established. That the λs-calculus translates so nicely into both the suspension calculus and the $\lambda\sigma$-calculus is a strong argument that the calculus is well-designed and natural for representing single substitutions.

Another point to note about the above theorem is that the suspension calculus may have to make multiple reading steps to simulate a single step in the λs-calculus. The primary reason for this comes from the (r3) and (r4) rules of the suspension calculus which are used to evaluate the result of applying a suspension to a de Bruijn index. Because a suspension can representing a substitution for various different de Bruijn indices, these rules must check to see which substitution applies for a given index. In the λs-calculus on the other hand, each closure represents a single substitution so when a de Bruijn index is encountered we can immediately check if it is the one being substituted for. This clearly gives a benefit in efficiency to the λs-calculus, but this benefit is not enough to offset the benefit gaining by merging substitutions [LNQ04].

4.1.3 The λs_e-calculus

The λs_e-calculus is an extension of the λs-calculus in order to gain confluence in a setting with graftable meta variables. The calculus achieves this by allowing what some call merging of substitutions, but what is more accurately described as permutation of substitutions. To be precise, the λs_e-calculus maintains the same syntax as the λs-calculus and extends the rewrite rules with the six rules in Figure 4.3.

Notice that each rule has careful restrictions on it to prevent looping behavior, but as shown in [Gui00] this is not enough: the λs_e-calculus fails to preserve strong normalization. Another more technical problem with the λs_e-calculus is that normal forms when in a context of graftable meta variables can become unwieldy. A λs_e-normal form has the same basic structure as a de Bruijn term, except that graftable meta variables can have sequences of closures and updates applied to them. The only restriction on these substitutions is that none of the s_e-rules apply to them [KR97]. This problem is brought to the forefront in the context of higher-order unification using the λs_e-calculus, where graftable meta variables

σ-σ-transition	$(a\,\sigma^i\,b)\,\sigma^j\,c \to (a\,\sigma^{j+1}\,c)\,\sigma^i(b\,\sigma^{j-i+1}\,c)$	if $i \leq j$
σ-φ-transition 1	$(\varphi^i_k\,a)\,\sigma^j\,b \to \varphi^{i-1}_k\,a$	if $k < j < k+i$
σ-φ-transition 2	$(\varphi^i_k\,a)\,\sigma^j\,b \to \varphi^i_k(a\,\sigma^{j-i+1}\,b)$	if $k+i \leq j$
φ-σ-transition	$\varphi^i_k(a\,\sigma^j\,b) \to (\varphi^i_{k+1}\,a)\,\sigma^j(\varphi^i_{k+1-j}\,b)$	if $j \leq k+1$
φ-φ-transition 1	$\varphi^i_k(\varphi^j_l\,a) \to \varphi^j_l(\varphi^i_{k+1-j}\,a)$	if $l+j \leq k$
φ-φ-transition 2	$\varphi^i_k(\varphi^j_l\,a) \to \varphi^{j+i-1}_l\,a$	if $l \leq k < l+j$

Figure 4.3: Additional rewrite rules for the λs_e-calculus

and normal forms play an important role in efficient unification procedures [ARK03].

4.2 A Calculus with Merging: the $\lambda\sigma$-calculus

The $\lambda\sigma$-calculus supports a general notion of composition of substitution walks and there exists a variant of it which is confluent in a setting with graftable meta variables [ACCL91, CHL96]. Unfortunately, Mellies was able to demonstrate that the calculus lacks preservation of strong normalization by presenting a simply typed lambda term for which an infinite $\lambda\sigma$-reduction path exists [Mel95].

We use the rest of this section to define the $\lambda\sigma$-calculus and construct translations to and from the suspension calculus.

Definition 4.2.1. *The syntax of $\lambda\sigma$-expressions is given by the following definition of terms, denoted a and b, and substitutions, denoted s and t.*

$$a \quad ::= \quad 1 \mid ab \mid \lambda a \mid a[s]$$
$$s \quad ::= \quad id \mid \uparrow \mid a \cdot s \mid s \circ t$$

The term $a[s]$ is called a *closure* and represents the term a with some substitution s to be applied to it. The substitution id is the identity substitution. The substitution $\uparrow$ is called *shift* and represents a increasing of all free de Bruijn indices by 1. The substitution $a \cdot s$ is called *cons* and represents a term a to be substituted for the first de Bruijn index along with a substitution s for the remaining indices. Lastly, the substitution $s \circ t$ represents the merging of the substitution s and t.

Note that the terms in this calculus only contain the first de Bruijn index. All others are represented by $1[\uparrow]$, $1[\uparrow \circ \uparrow]$, $1[(\uparrow \circ \uparrow) \circ \uparrow]$ etc. We abbreviate the n^{th} de Bruijn index as $1[\uparrow^{n-1}]$. With this in mind, the rules for the $\lambda\sigma$-calculus are presented in Figure 4.4.

(Beta)	$(\lambda a)b \to a[b \cdot id]$		
(App)	$(a\ b)[s] \to a[s]\ b[s]$	(VarId)	$1[id] \to 1$
(Abs)	$(\lambda a)[s] \to \lambda a[1 \cdot (s \circ \uparrow)]$	(VarCons)	$1[a \cdot s] \to a$
(Clos)	$a[s][t] \to a[s \circ t]$	(IdL)	$id \circ s \to s$
(Map)	$(a \cdot s) \circ t \to a[t] \cdot (s \circ t)$	(ShiftId)	$\uparrow \circ\, id \to \uparrow$
(Ass)	$(s \circ t) \circ u \to s \circ (t \circ u)$	(ShiftCons)	$\uparrow \circ\, (a \cdot s) \to s$

Figure 4.4: Rewrite rules for the $\lambda\sigma$-calculus

4.2.1 Suspension Expressions to $\lambda\sigma$-expressions

The translation from suspension expressions to $\lambda\sigma$-expressions works by unfolding the information which is represented by the indices and embedding levels of the suspension calculus into individual substitution operations of the $\lambda\sigma$-calculus. Accounting for this, the rest of the translation is straightforward and translates suspension expressions into corresponding $\lambda\sigma$-expressions: suspension to closure, nil to id, cons to cons, and merged to merged. Besides the difference in representing renumberings, the syntax of the two calculi match up nicely.

Definition 4.2.2. *The translation S from suspension terms to $\lambda\sigma$-terms and the translation R from pairs of a suspension environment and a new embedding level to $\lambda\sigma$-substitutions are defined simultaneously by recursion as follows:*

1. *For a term t, $S(t)$ is 1 if t is $\#1$, $1[\uparrow^n]$ if t is $\#(n+1)$ with $n \geq 1$, $(S(a)\ S(b))$ if t is $(a\ b)$, $\lambda\, S(a)$ if t is $\lambda\, a$, and $S(t')[R(e, nl)]$ if t is $[\![t, ol, nl, e]\!]$.*

2. *For an environment e and natural number j, $R(e, j)$ is $(\ldots(id \overbrace{\circ \uparrow) \circ \uparrow) \circ \ldots)}^{j}$ if e is nil, $(\ldots((S(t) \cdot R(e', n)) \overbrace{\circ \uparrow) \circ \uparrow) \circ \ldots)}^{j-n}$ if e is $(t, n) :: e'$, and $R(e_1, nl_1) \circ R(e_2, j - (nl_1 \mathbin{\dot{-}} ol_2))$ if e is $\{\!\{e_1, nl_1, ol_2, e_2\}\!\}$.*

The translation from suspension expressions to $\lambda\sigma$-expressions includes a translation $R(e, j)$ which translates the environment e relative to the embedding level j. Restrictions must be placed on this translation to ensure the definition is well-formed. For example, looking at the second case for $R(e, j)$, one might worry that $j < n$ in which case having $(j - n)$ shifts does not make sense. We can ensure this never happens by requiring that

$lev(e) \leq j$ every time $R(e, j)$ is called. Enforcement of this is provided by the wellformedness properties of suspension terms.

Theorem 4.2.1. *If t is a suspension term then $S(t)$ is well-defined.*

Proof. The property must be proved simultaneously with the property that if e is a suspension environment and j is an integer with $j \geq lev(e)$ then $R(e, j)$ well-defined. □

Due to occurrences of the identity substitution and small differences in associativity, the $\lambda\sigma$-calculus does not simulate the suspension calculus. Instead, we show that normal forms are preserved by the translation.

Theorem 4.2.2. *Let a and b be suspension terms such that $a \rhd_{rm} b$. Then there exists a $\lambda\sigma$-term t such that $S(a) \rhd_{\sigma}^{*} t$ and $S(b) \rhd_{\sigma}^{*} t$.*

Proof. The proof uses the dual property that if a and b are suspension environments and j is an integer such that $j \geq lev(a)$ then $R(a, j) \rhd_{rm}^{*} R(b, j)$. We can then prove both properties by case analysis on the rule used to transition from a to b. □

Finally, we argue that S is information preserving by showing that it is one-to-one. Note that this property is quite strong since we are translating from a calculus with merging to another calculus with merging. Before, when we translated from a calculus without merging to one with merging, this property was more obvious since only the substitution representations might overlap. Here we must worry about substitution representations and also merged substitution representations.

Theorem 4.2.3. *The translation S is one-to-one.*

Proof. The result follows easily from noticing that $R(e, j)$ can never equal $\uparrow^{k}$ for any e, j, and k. □

Because the $\lambda\sigma$-calculus and the suspension calculus seem to be equally expressive we can define a translation in the other direction in the following section.

4.2.2 $\lambda\sigma$-expressions to Suspension Expressions

The translation from $\lambda\sigma$-expressions to suspension expressions proceeds in the obvious way except for a special case when translating $s \circ \uparrow$ which changes the shift substitution into the corresponding renumbering concept expressed in embedding levels and indices.

Definition 4.2.3. *The translation T from $\lambda\sigma$-terms to suspension terms and the translation E from $\lambda\sigma$-substitutions to triples of an old embedding level, a new embedding level, and a suspension environment are defined simultaneously by recursion as follows:*

1. *For a term t, $T(t)$ is $\#1$ if t is 1, $\#(n+1)$ if t is $1[\uparrow^n]$, $(T(a)\ T(b))$ if t is $(a\ b)$, $\lambda\, T(a)$ if t is $\lambda\, a$, and $[\![T(a), ol, nl, e]\!]$ if t is $a[s]$ where $(ol, nl, e) = E(s)$.*
2. *For a substitution s, $E(s)$ is $(0, 0, nil)$ if s is id, $(0, 1, nil)$ if s is $\uparrow$, $(ol+1, nl, (T(a), nl) :: e)$ if s is $a \cdot s'$ where $(ol, nl, e) = E(s')$, $(ol, nl+1, e)$ if s is $s' \circ \uparrow$ where $(ol, nl, e) = E(s')$, and $(ol_1 + (ol_2 \dot{-} nl_1), nl_2 + (nl_1 \dot{-} ol_2), \{\!\{e_1, nl_1, ol_2, e_2\}\!\})$ if s is $s_1 \circ s_2$ where $(ol_1, nl_1, e_1) = E(s_1)$ and $(ol_2, nl_2, e_2) = E(s_2)$.*

If more than one case apply to the same expression, we require that first one listed is the one used.

Theorem 4.2.4. *For every $\lambda\sigma$-term a, $T(a)$ is a well-formed suspension term.*

Proof. The proof is by induction using the dual property that for every $\lambda\sigma$-substitution s such that $(ol, nl, e) = E(s)$ we have $ol = len(e)$, $nl \geq lev(e)$, and e is a well-formed suspension environment. □

The suspension calculus is not capable of simulating the $\lambda\sigma$-calculus and this is not a bad property. If the suspension calculus were able to simulate the $\lambda\sigma$-calculus, then the suspension calculus would be able to simulate the Mellies counterexample to preservation of strong normalization [Mel95]. The primary reason for the lack of simulation is the Ass rule which establishes an association rule for merged substitutions. In the suspension calculus, however, such association is a property we have proven with significant work, see Section 3.8.1, but it is not a rule of the calculus. Instead of showing simulation, we show normal forms are preserved by the translation.

Theorem 4.2.5. *Let a and b be $\lambda\sigma$-terms such that $a \rhd_\sigma b$. Then there exists a suspension-term t such that $T(a) \rhd_{rm}^* t$ and $T(b) \rhd_{rm}^* t$.*

Proof. A naive approach to this theorem would be filled with special cases to account for the special cases present in the translations T and E. In order to avoid this note that in the special case of $T(1[\uparrow^n]) = \#(n+1)$ if we had used the more general translation of $a[s]$ we would have produced $[\![\#1, 0, n, nil]\!]$. Since these two terms are $\rhd_{rm}^*$-convertible we can pick the second one for this theorem and ignore the special case. The same result holds for the special case of $E(s \circ \uparrow)$.

The other difficulty in proving this theorem is that we will need a corresponding property for $\lambda\sigma$-substitutions. Naively, this property might be that if $s \rhd_\sigma t$ then the old and new embedding level components of $E(s)$ and $E(t)$ are equal and the environment components rewrite to a common environment. This will fail because of the (Map) rule in the $\lambda\sigma$-calculus which has the form $(a \cdot s_1) \circ s_2 \rightarrow a[s_2] \cdot (s_1 \circ s_2)$. Letting $t = T(a)$, $(ol_1, nl_1, e_1) = E(s_1)$, and $(ol_2, nl_2, e_2) = E(s_2)$, the environment components of the translation E applied to the left and right sides of the (Map) rule are $\{\!\{(t, nl_1) :: e_1, nl_1, ol_2, e_2\}\!\}$ and $([\![t, ol_2, nl_2, e_2]\!], nl_2 + (nl_1 \dot{-} ol_2)) :: \{\!\{e_1, nl_1, ol_2, e_2\}\!\}$, respectively. Note that this is very similar to our rule (m6) but different in that e_2 might not have the form $(s, l) :: e_2'$ and also we use nl_2 instead of the level of e_2. Because of these problems, these two environments are not rewritable to a common environment. Instead, we generalize the property to state that the environment components should be rewritable to similar environments, see Section 3.11, from which the result follows. □

The translation T is not one-to-one because the $\lambda\sigma$-substitutions $\uparrow$ and $id \circ \uparrow$ translate to the same tuple. We can, however, prove that this translation is a left-inverse of the translation S from suspension term to $\lambda\sigma$-terms. Because the translation S is information preserving, this result is strong evidence that T is also information preserving. Moreover, this result shows that our translations are well balanced and therefore, hopefully natural.

Theorem 4.2.6. *For every suspension term t, $T(S(t)) = t$.*

Proof. The proof is by induction using the dual property that for every suspension environment e and integer j such that $j \geq lev(e)$, we have $E(R(e, j)) = (ol, j, e)$ where $ol = len(e)$. □

Chapter 5

Preservation of Strong Normalization

Normal forms hold a special place in the lambda calculus. The normal form of a term has the same meaning as the original term without any β-contraction work left. This makes normal forms an ideal basis for unification procedures over the lambda calculus. By focusing on normal forms, such procedures can ignore the β-contraction aspect of the lambda calculus, and instead focus just on the binding structure of normal forms.

Because of this lofty position, much work has been put into determining when normal forms exist and how to compute them. For instance, a strong motivation behind the simply typed lambda calculus in Section 2.3.2 is that normal forms are guaranteed to exist for all terms in the calculus. Such terms are called *normalizable*. In fact, any reduction of a term from that calculus is finite and must reach the normal form, prompting the title *strongly normalizable*. This is not always the case for other variants of the lambda calculus. In those instances, a reduction strategy must be carefully chosen which reduces a term to its normal form, provided one exists. A well known strategy which has this behavior is called *normal order* reduction and consists of always contracting the leftmost outermost β-redex, which we will call the *leading redex*. The fundamental reason why this strategy works is that contracting anything other than leading redex will not affect the existence of the leading redex. Thus the leading redex will persist forever unless we contract it, and therefore we choose to contract it first. In this chapter we look at generalizations of these notions of normalizability, strong normalizability, and reduction strategies to the context of explicit substitution calculi.

5.1 Preservation of Normalizability

Explicit substitution calculi are elaborations of the lambda calculus and as such they preserve normalizability. That is, given a normalizable term in the lambda calculus, it remains normalizable in an explicit substitution calculus. The reason is is that each β-contraction step in the lambda calculus can be matched in an explicit substitution calculus by a simulated β-contraction step followed by a series of substitution steps. This approach to computing normal forms is logically sound, but it removes the practical benefits of using an explicit substitution calculus. We can make two improvements on it.

The first improvement is actually one which can be realized in the lambda calculus. Because we are often interested in finding normal forms for the purpose of unification, we can develop a weaker notion of normal form which does not require as much work to compute, but still suffices for the purpose of unification. Such a notion is capture by reducing terms to the form $(\lambda\,\lambda\,\ldots\,\lambda\,(h\ t_1\ \ldots\ t_n))$ where h is either a constant, a de Bruijn index, or a meta variables. This is called a *head normal form*. Another term in head normal form then unifies with this one if and only if it has the form $(\lambda\,\lambda\,\ldots\,\lambda\,(h\ s_1\ \ldots\ s_n))$ where the number of leading lambdas is the same and t_i unifies with s_i for $1 \leq i \leq n$. This method saves us from having to normalize the terms $t_1, \ldots, t_n, s_1, \ldots, s_n$ in the case that unification fails. A simple and complete procedure for computing head normal forms is to perform normal order reduction until a head normal form is reach and then stop there. This is called *head normalization* and in this case the leading redex is referred to as the *head redex*.

The second improvement we can make is to generalize the notion of head normalization to the explicit substitution context. Nadathur has made such a generalization in the case of the suspension calculus and has proven that such a procedure always finds the normal form of a normalizable term [Nad99]. The idea behind Nadathur's notion of head reduction is to define a head redex as either the head β-redex or a $\rhd_{rm}$-redex which occurs above the head β-redex. Generalized head reduction then consists of contracting head redexes until no more head reductions are possible. This generalization provides more freedom in computing head normal forms, but it retains the essential property of head reduction: that a head normal form is always reached in a finite number of steps, assuming one exists. To see why this is true, notice that the reading and merging rules are terminating, so any infinite reduction would have to consist of infinitely many (β_s) applications on head redexes. But we can map each of these applications to the contraction of the corresponding β-redex in the lambda calculus. This is as simple as taking the $\rhd_{rm}$-normal form before and after applying (β_s). Thus any infinite generalized head reduction sequence in the suspension calculus can be mapped onto an infinite head reduction sequence in the lambda calculus.

5.2 Preservation of Strong Normalization

A more complicated issue is whether a strongly normalizing term in the lambda calculus remains strongly normalizing within an explicit substitution calculus. A calculus is said to have *preservation of strong normalization* (PSN) if this is the case for every strong normalizing term. This is a desirable property because it speaks to the coherence of the calculus. Intuitively one might expect this property to always hold since explicit substitution calculi are elaborations of the lambda calculus, but this is not the case. The interaction of contraction and substitution rules in explicit substitution calculi creates the possibility of reduction sequences which do not correspond to reductions in the lambda calculus. The existence of such reduction sequences speaks poorly for the structure of an explicit substitution calculus. In the remainder of this chapter, we focus on the issue of preservation of strong normalization in various explicit substitution calculi.

5.3 PSN in Calculi without Substitution Interaction

In calculi without rules for interactions between substitutions, preservation of strong normalization is usually true. In such calculi, substitutions are generated by a step which simulates β-contraction and then those substitutions are pushed down through the term until they can be evaluated. Because of this essentially linear path, it is very easy to take an arbitrary substitution and determine which β-contraction generated it. By connecting these β-contractions in the explicit substitution calculus to β-contractions in the lambda calculus, any infinite reduction sequence in the former can be mapped onto one in the latter.

To take an example, we consider here the proof of preservation of strong normalization for the λs-calculus (see Section 4.1.2). In this setting a closure refers to a term of the form $a\,\sigma^i\,b$ and the inside of a closure refers to the term b. Suppose we have an infinite reduction of some term in this calculus. We know that the s rules of the λs-calculus are strongly normalizing, so the infinite reduction must contain infinitely many contractions of β-redexes using the σ-generation rule. At each step of this infinite reduction, we can look at the s-normal form of the current term to see what progress is being made with respect to the lambda calculus. Clearly each step which uses an s rule does not change the s-normal form. For the σ-generation steps, some may change the s-normal form and some may leave it the same. Those that change the s-normal form correspond to β-contractions in the lambda calculus. If there are an infinite number of such steps then we can use the s-normal forms as an infinite reduction sequence in the lambda calculus. The other possibility is that only finitely many σ-generation steps correspond to changes in the s-normal form. Now any σ-generation step which occurs at the top level (outside of any closures) will be one of these steps which changes the s-normal form, and therefore only finitely many of our σ-generation steps occur at the top level. Because there are only finitely many such steps, we can find a point in our infinite reduction at which all σ-generation steps occur inside of closures. By the infinite pigeonhole principle, there must be one closure which contains an infinite reduction inside it. Thus we have reduced our infinite reduction sequence to an infinite reduction which occurs entirely within a single closure.

The next key step in the proof is to trace each closure back to the β-redex from which it was created. This is possible since only the σ-generation rule can create closures. Using this idea we can take our infinite reduction which occurs within some closure, say $a\,\sigma^i\,b$, and know that it came from a term of the form $((\lambda\,a')\ b')$ where b' rewrites to b. Now instead of contracting the β-redex we can follow the infinite reduction path which exists for b'. Because this b' is no longer inside of a closure, the σ-generation steps inside it will correspond to reductions in the lambda calculus for the s-normal form. By the same reasoning we have followed so far, what must occur is that this b' eventually generates a closure which contains an infinite reduction. But this closure can again be unwound and mapped into a reduction sequence in the lambda calculus. By repeating this process indefinitely we generate an infinite reduction sequence in the lambda calculus.

5.4 Problems for Calculi with Substitution Interaction

Some calculi have rules of interactions between substitutions, the nature of which depend on the motivation for including them in the calculus. One motivation for substitution interaction is to regain confluence in a setting with graftable meta variables. For instance, in the λs-calculus, consider the term $((\lambda\,X)\ b)\,\sigma^i\,c$ where X is a graftable meta variable and b and c are arbitrary terms. On the one hand we can contract the redex to obtain $(X\,\sigma^1\,b)\,\sigma^i\,c$, while on the other we can first distribute the substitution and then perform the reduction to generate $(X\,\sigma^{i+1}\,c)\,\sigma^1(b\,\sigma^i\,c)$. These two terms cannot be reduced to a common term because X is a graftable meta variable. In order to fix this, the λs_e-calculus, see Section 4.1.3, extends the λs-calculus and introduces rules for interactions between substitutions [KR97]. One of these rules deals exactly with the case we have,

$$\sigma\text{-}\sigma\text{-}transition \qquad (a\,\sigma^i\,b)\,\sigma^j\,c \to (a\,\sigma^{j+1}\,c)\,\sigma^i(b\,\sigma^{j-i+1}\,c) \qquad \text{if } i \leq j$$

Applying this rule reconciles the two reductions into the term $(X\,\sigma^{i+1}\,c)\,\sigma^1(b\,\sigma^i\,c)$.

The danger in admitting a permutation rule is that once we have permuted two substitutions, we might try to permute them again. The λs_e-calculus tries to avoid this situation by placing side conditions on the permutation rules so that one substitution can only be permuted inside another if the outside one represents a contraction from higher up in the term than the inner substitution. For example, in a term of the form $((\lambda \ldots (\lambda\, a)\ b \ldots)\ c)$ we allow the contraction of the outer redex to be permuted inside the contraction of the inner redex. An additional wrinkle, however, is that we must also add rules for interactions with updating functions. It is exactly these additional interaction rules which causes the λs_e-calculus to lose preservation of strong normalization, as proved by Guillaume [Gui00].

Guillaume and David solved this problem by introducing the λ_{ws}-calculus which replaces update functions with labels representing the renumbering to be done [DG01a]. These labels are then part of the normal forms of terms in the calculus since there are no rules for propagating their effects. Thus this calculus corresponds to a version of the λs_e-calculus where restrictions are placed on the ability to propagate updating functions. These restrictions are not so severe that confluence in presence of graftable meta variables is lost. Furthermore, with these restrictions, Guillaume and David are able to show that if a term has an infinite reduction sequence then contracting and propagating a leading redex preserves that infinite reduction sequence. Using this they map every infinite reduction in the λ_{ws}-calculus into an infinite normal order reduction sequence in the lambda calculus, thus proving preservation of strong normalization. The key to using the leading redex is that it is above every other substitution which may be encountered during propagation. Thus it can be permuted inside of those substitutions without disturbing their effect on the infinite reduction.

A different approach to substitution interaction is to allow full combination of substitutions such as in the suspension calculus and the $\lambda\sigma$-calculus [ACCL91]. The benefit of this approach is that the resulting calculus is often very efficient for direct implementation. Additionally, the merging rules for these calculi are usually strong enough that confluence in the presence of graftable meta variables can be recovered. The downside of allowing merging is that it creates new interaction possibilities for substitutions, and this may lead to the loss of preservation of strong normalization. Such is the case in in the $\lambda\sigma$-calculus where Mellies demonstrated a strongly normalizing term along with an infinite $\lambda\sigma$-reduction [Mel95].

The essential problem with merging in the $\lambda\sigma$-calculus is that superfluous terms can be generated and then allowed to interact with other substitutions. For instance in a substitution of the form $\uparrow \circ (a \cdot s)$ we know that the term a is going to be eventually pruned by the shift. However if we have an outer substitution applied to this substitution, $(\uparrow \circ (a \cdot s)) \circ r$, then we can apply the association rule for merged environments to rewrite this term to $\uparrow \circ ((a \cdot s) \circ r)$. From here we can map the substitution r onto a to yield $a[r]$ which is again superfluous, but may contain significant reduction work.

This idea is played out in completely in the Mellies counterexample which begins with a term of the form $((\lambda\, a)\ b)[s]$ and rewrites it so that the substitution s is able to interact with a version of itself.

$$\begin{aligned}((\lambda\, a)\ b)[s] &\rhd_{App} (\lambda\, a)[s]\ b[s]\\ &\rhd_{Abs} \lambda\, a[1\cdot(s\circ\uparrow)]\ b[s]\\ &\rhd_{Beta} a[1\cdot(s\circ\uparrow)][b[s]\cdot id]\\ &\rhd_{Clos} a[(1\cdot(s\circ\uparrow))\circ(b[s]\cdot id)]\\ &\rhd_{Map} a[1[b[s]\cdot id]\cdot((s\circ\uparrow)\circ(b[s]\cdot id))]\\ &\rhd_{Ass} a[1[b[s]\cdot id]\cdot(s\circ(\uparrow\circ(b[s]\cdot id)))]\end{aligned}$$

From here on we can focus solely on the substitution $s\circ(\uparrow\circ(b[s]\cdot id))$. Notice that at this point the $b[s]$ component here is vacuous. Because of the $\uparrow$, the $b[s]$ should be removed as soon as we apply ShiftCons. Unfortunately, the rules of the $\lambda\sigma$-calculus allow us to play with this vacuous term and produce an infinite sequence. If we consider that s might be of the form $((\lambda\, a)\ b)\cdot id$ and if we abbreviate $(\uparrow\circ(b[s]\cdot id))$ as s' then we can rewrite the term $s\circ(\uparrow\circ(b[s]\cdot id))$ as follows.

$$\begin{aligned}s\circ(\uparrow\circ(b[s]\cdot id)) = (((\lambda\, a)\ b)\cdot id)\circ s' &\\ &\rhd_{Map} ((\lambda\, a)\ b)[s']\cdot(id\circ s')\\ &\rhd_{IdL} ((\lambda\, a)\ b)[s']\cdot s'\\ &\rhd_{App} ((\lambda\, a)[s']\ b[s'])\cdot s'\\ &\rhd_{Abs} (\lambda\, a[1\cdot(s'\circ\uparrow)]\ b[s'])\cdot s'\\ &\rhd_{Beta} (a[1\cdot(s'\circ\uparrow)][b[s']\cdot id])\cdot s'\\ &\rhd_{Clos} (a[(1\cdot(s'\circ\uparrow))\circ(b[s']\cdot id)]\cdot s'\\ &\rhd_{Map} a[1[b[s']\cdot id]\cdot((s'\circ\uparrow)\circ(b[s']\cdot id))]\cdot s'\\ &\rhd_{Ass} a[1[b[s']\cdot id]\cdot(s'\circ(\uparrow\circ(b[s']\cdot id)))]\cdot s'\end{aligned}$$

Here we again have a subterm of the form $s'\circ(\uparrow\circ(b[s']\cdot id))$. Using this we can repeat the above reasoning to produce an infinite sequence.

5.5 Status of PSN for the Suspension Calculus

Preservation of strong normalization for the suspension calculus is an open problem. In this section we explain why the counterexample from the $\lambda\sigma$-calculus and the proof techniques of the λs-calculus are insufficient in resolving this problem.

To start, consider how the counterexample from the $\lambda\sigma$-calculus would proceed in the suspension calculus. The term $((\lambda\, a)\ b)[s]$ in the $\lambda\sigma$-calculus corresponds to a term $[\![(\lambda\, a)\ b, ol, nl, e]\!]$. Then the reduction can proceed as follows.

$$
\begin{aligned}
&[\![(\lambda\, a)\ b, ol, nl, e]\!] \\
&\qquad \rhd_{(r5)} [\![\lambda\, a, ol, nl, e]\!]\ [\![b, ol, nl, e]\!] \\
&\qquad \rhd_{(r6)} (\lambda\, [\![a, ol+1, nl+1, (\#1, nl+1) :: e]\!])\ [\![b, ol, nl, e]\!] \\
&\qquad \rhd_{(\beta_s)} [\![[\![a, ol+1, nl+1, (\#1, nl+1) :: e]\!], 1, 0, ([\![b, ol, nl, e]\!], 0) :: nil]\!] \\
&\qquad \rhd_{(m1)} [\![a, ol+1, nl, \{\!\{(\#1, nl+1) :: e, nl+1, 1, ([\![b, ol, nl, e]\!], 0) :: nil\}\!\}]\!] \\
&\qquad \rhd_{(m6)} [\![a, ol+1, nl, ([\![\#1, 1, 0, ([\![b, ol, nl, e]\!], 0) :: nil]\!], nl) :: \\
&\qquad\qquad\qquad\qquad \{\!\{e, nl+1, 1, ([\![b, ol, nl, e]\!], 0) :: nil\}\!\}]\!] \\
&\qquad \rhd_{(m5)} [\![a, ol+1, nl, ([\![\#1, 1, 0, ([\![b, ol, nl, e]\!], 0) :: nil]\!], nl) :: \{\!\{e, nl, 0, nil\}\!\}]\!] \\
&\qquad \rhd_{(m2)} [\![a, ol+1, nl, ([\![\#1, 1, 0, ([\![b, ol, nl, e]\!], 0) :: nil]\!], nl) :: e]\!]
\end{aligned}
$$

At this point we can focus on the first term in the environment and reduce it as follows.

$$
\begin{aligned}
[\![\#1, 1, 0, ([\![b, ol, nl, e]\!], 0) :: nil]\!] &\rhd_{(r3)} [\![[\![b, ol, nl, e]\!], 0, 0, nil]\!] \\
&\rhd_{(m1)} [\![b, ol, nl, \{\!\{e, nl, 0, nil\}\!\}]\!] \\
&\rhd_{(m2)} [\![b, ol, nl, e]\!]
\end{aligned}
$$

This leaves the original term as $[\![a, ol+1, nl, ([\![b, ol, nl, e]\!], nl) :: e]\!]$. There doesn't appear to be any means for an infinite reduction from this since we don't have the environment e acting on itself, as was the case in the $\lambda\sigma$-calculus. Instead let us reconsider the steps we took in producing this term and ask if we could have chosen a different reduction path once we merged the two environments. The answer is that there can be no other reduction path for this merged environment in this case or for any merged environment in the general case. Looking at the rules which operate on merged environments, we see that there are no choices in which rule can be applied at a given stage except for the trivial overlap between (m2) and (m3). Thus the merging process is deterministic. Furthermore, given an environment of the form $[\![e_1, nl_1, ol_2, e_2]\!]$, there are no rules which allow the outside context of this environment to have an effect on e_1 or e_2, until after the merging is performed. In this way, the merging process of the suspension calculus can be viewed as an atomic action. We can conceivably imagine replacing (m1)-(m6) with a single merging rule which rewrites a term of the form $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!]$ to one of the form $[\![t, ol', nl', e']\!]$ where e' is a simple environment. The benefit of the separate rules (m1)-(m6) is that this large merging operation is done in a lazy fashion.

On the other hand, we can think of extending the proof of preservation of strong normalization for the λs-calculus to apply to the suspension calculus. In the λs-calculus we are able to trace each closure back to the β-redex which created it, because only the β-contraction rule can generate closures. In the suspension calculus, environment terms can be created either by β-contraction or by merging of substitutions. Thus tracing an environment term back to a single β-redex in the lambda calculus is extremely difficult. Furthermore, the λs-calculus had a fairly linear order in generating and propagating substitutions, while the suspension calculus has the possibility that propagating one substitution might mean merging with other substitutions along the way. This allows the suspension calculus more much freedom in choosing reduction paths, but it also makes mapping those reduction paths onto the lambda calculus significantly more difficult.

Chapter 6

Conclusion

In this thesis we have presented a version of the suspension calculus which combines the desirable theoretical properties of the original suspension calculus with the practical benefits of the derived suspension calculi. This new version is created not by adding more to the calculus, but by simplifying what is already there. This simplification has the additional benefit of rationalizing the structure of the calculus, making it possible to easily superimpose additional logical structure over it. We have illustrated this capability by showing how typing in the lambda calculus can be treated in the resulting framework and by presenting a natural translation into the $\lambda\sigma$-calculus. We have also shown how the substitution mechanism supports combination of substitution walks while remaining confluent and terminating. Building on this, we have proven that the full suspension calculus is confluent even in the presence of graftable meta variables. The question of preservation of strong normalization relative to the suspension calculus remains open. However, we conjecture that it is true and we have presented arguments as to why this belief might be correct. If this property is indeed true then it would make the suspension calculus the only explicit substitution calculus which possesses the three properties deemed to be most desirable.

Another contribution of this thesis is a survey of the realm of explicit substitution calculi. We have utilized the suspension calculus in this process. In particular, we have described translations between other popular calculi and the suspension calculus towards understanding and contrasting their relative capabilities. To give substance to this approach, we have established relevant properties of the translations such as their correctness and their ability to preserve important computational properties of the calculi they relate.

This thesis would be incomplete without a discussion of the possible ways of building on the results it presents.

Preservation of Strong Normalization

Preservation of strong normalization is a problem of significant theoretical interest because it speaks to the coherence of the calculus. As already mentioned, we believe that it can be proven true in the case of the suspension calculus. The basis of this belief is that the merging (or permutation) of substitutions which has caused the property to fail in other calculi is handled correctly in the suspension calculus. Whereas the $\lambda\sigma$-calculus, the only other calculus that allows combination of substitutions, allows us to make choices in how to unravel substitution combination, the suspension calculus treats substitution combination as a deterministic and pseudo-atomic function. While this intuition appears to be accurate, working it out into a proof has been difficult. In particular, reflecting an arbitrary reduction sequence in the suspension calculus into one in the lambda calculus appears complicated but this seems to be necessary to show that infinite sequences in first context must be matched by infinite ones in the second. Nevertheless, we believe that the simpler set of combination rules gives us a better handle on this matter and hence are hopeful of using it to construct an actual argument.

Higher-Order Unification using the Suspension Calculus

As mentioned in the introduction, one benefit of using explicit substitutions is that it allows substitution notions to be actively used by processes that operate on the lambda calculus such as unification. Recent work has exploited this feature in producing a higher-order unification procedures based on a variant of the $\lambda\sigma$-calculus which supports graftable meta variables [DHK95]. The original suspension calculus also supports graftable meta variables and so this unification idea could have been worked out in its context as well. However, the incentive for doing this has been small because the complexity of its combination rules limits the benefit of doing this in actual implementations. Derived calculi based on the suspension calculus simplify these combination rules into a couple rules which are useful for head normalization, but these calculi are not confluent when graftable meta variables are added. By contrast, the suspension calculus presented in this thesis has the property of confluence even in the presence of graftable meta variables and also has a collection of combination rules that is simple enough to use directly in an implementation. The benefit of developing the new approach to unification based on the suspension calculus is that it treats renumbering in a more efficient manner than the $\lambda\sigma$-calculus and so a higher-order unification procedure based on the suspension calculus is likely to have better behavior in practice.

Compilation of Strong Reduction

Functional programming languages use a notion of reduction where an expression that has a top level abstraction is treated as a value. This form of reduction, where it is unnecessary to look underneath abstractions, is called *weak reduction*. Weak reduction is easily performed in an interpretive setting by keeping an environment which tracks variable bindings. It is also possible to compile weak reduction and the Categorical Abstract Machine which underlies the Objective Caml programming language provides a framework for doing exactly this [CCM87]. In the representational use of the lambda calculus it may be necessary to compare underneath lambda abstraction leading to the need to perform reductions even in such contexts. This is called *strong reduction*. Explicit substitution calculi provide a basis for realizing strong reduction and in fact an interpreted approach has been developed based on the suspension calculus and used in a λProlog implementation [LNQ04]. An approach to using a compilation based realization of strong reduction has also been described in the context of the Coq system [GL02]. However, this approach is somewhat ad-hoc and is based on repeated calls to the reduction machinery underlying the categorical abstract machine. We believe a uniform compilation model can be developed using an explicit substitution notation such as the suspension calculus.

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