Preconditionning techniques for the solution of the Helmholtz equation by the finite element method *

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Abstract. This paper discusses 2D solutions of the Helmholtz equation by finite elements. It begins with a short survey of the absorbing and transparent boundary conditions associated with the DtN technique. The solution of the discretized system by means of a standard Galerkin or Galerkin Least-Squares (GLS) scheme is obtained by a preconditioned Krylov subspace technique, specifically a preconditioned GMRES iteration. The stabilization paremeter associated to GLS is computed using a new formula. Two types of preconditioners, ILUT and ILU0, are tested to enhance convergence.

Key words: Helmholtz Equation, acoustic scattering, DtN technique, Finite element method, GMRES iterative method, Incomplete factorization, ILUT, and ILU0.

1 Introduction

In the harmonic regime most diffraction phenomena are governed by the Helmholtz equation. Propagation/scattering problems are often defined over open (non- bounded) domains and are, as a result, solved by the boundary element method. In this case only the external surface is discretized. This leads to systems of equations of relatively small dimensions which are, however, dense. For 3-D problems in which relatively high frequencies come into play, the required memory and computational resources can quickly exceed those afforded by available workstations. For these reasons and also because integral-based techniques are restricted to linear and isotropic problems, the finite element method is currently enjoying a regain of interest.

The use of the finite element method requires that we define the boundary of the discretized domain. From a practical point of view, the actual obstacle is surrounded by an artificial boundary, located at a finite distance from the external surface. The scattered field outside the computational domain is thus represented either by boundary conditions known as 'absorbing' (ABC), which are specified on this boundary, or by infinite elements.

* This research has been funded by the National Sciences and Engineering Research Council of Canada (NSERC), by the Army Research Office under grant DAAD19-00-1-0485, and by NSF. Support was also provided by the Minnesota Supercomputing Institute.

In both cases, the idea is to prevent the reflection of waves by the artificial boundary. Berenger [3] proposed to replace the boundary by an absorbing layer, or PML (Perfectly Matched Layer), of finite width, whose role is precisely to damp the waves diffracted by the obstacle. The sizes of the computational domain are thus increased by the width of the PML.

The technique referred to as DtN (Dirichlet to Neumann), and introduced by Givoli and Keller [8], can be viewed as a general procedure for specifying exact boundary conditions, known as 'transparent' (TBC), in the case of artificial boundaries of simple geometric shapes (circle, ellipse, sphere, ellipsoid). Since the TBC is non local, it leads to couplings between all degrees of freedom at the artificial boundary, which may entail excessive memory requirements. To overcome these difficulties, local absorbing boundary conditions, such as those of Robin, have been developed. This simple condition is easy to implement but gives poor results.

Enquist and Majda [5] proposed an approach which consists of approximating the DtN operator. Bayliss and al. [1] developed an asymptotic solution outside the domain to establish these conditions. However, the local ABC cannot entirely eliminate the parasitic reflections. On the numerical side, the solution of the Helmholtz equation is particularly difficult when the frequencies involved are high, because of the loss of the elliptic character, and because of the oscillatory behavior of its solution. The meshes must therefore be very fine in the case of a standard discretization scheme in order to minimize numerical noise. Special discretization schemes, whose implementations are rather involved, have been developed to circumvent these difficulties.

The system of equations obtained from the Galerkin discretization scheme is sparse, complex and symmetric (but non-Hermitian). It is also generally not diagonal dominant and its Hermitian part is not positive definite. The iterative method GMRES [13], combined with an efficient preconditioner was found to be fairly robust for solving systems of this type, especially when the frequencies involved are high. Indeed, direct methods become exceedingly expensive both in terms of memory and computations when solving very large size systems. Industrial applications often lead to the solution of systems of equations of several millions of unknowns. When attempting to convert this complex system into a real one the number of unknowns doubles, though each of the unknowns is now real and occupies half the space occupied by a complex number. What is more serious is the impact on the preconditioner. Converting the complex problem into a real one in the standard way amounts to reordering the complex data and the resulting system becomes more difficult to precondition. In this case the preconditioned GMRES algorithm converges in a reasonable time only when a full factorization of a Crout type [15] is performed. This factorization remains expensive. Numerical analysts are currently devoting enormous efforts to develop effective preconditioners. However, their methods have had limited success for highly indefinite problems.

In this work we propose to investigate the usefulness of a complex version of ILUT (Incomplete LU factorization with Threshold), developed by Saad [11]. This preconditioner is derived from direct solution methods. The basic idea is to apply an incomplete factorization of the type LU, with reduced cost, to the original system of equations resulting from the discretization of the Helmholtz equation.

2 Mathematical model

We are interested in diffraction of an acoustic wave originating from infinity on a bounded obstacle with boundary Γ. The wave propagates in an open medium Ω_e. The objective is to develop a model using the finite element method to deal with problems of acoustic diffraction. First we present the boundary value problem which governs these phenomena in unbounded media. This problem is not adapted to a numerical solution by the finite

element method. It is reformulated by invoking the DtN technique, into another problem which is better adapted to this type of solution. The radiation condition at infinity is thus replaced by a particular boundary condition on an artificial boundary.

2.1 General model

The problem to solve on the open domain Ω_e is as follows:

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \quad \Omega \\ u = g \text{ on } \quad \Gamma \\ \lim_{r\to\infty} \sqrt{r}\ \left(\frac{\partial u}{\partial \boldsymbol{n}} - iku\right) = 0 \end{cases} \tag{1}$$

where u and f denote, respectively, the wave diffracted by Γ and a source function such that $f = 0$ outside the artificial boundary, which typically assumes a circular shape. The second equation of the above problem is the Sommerfeld radiation condition. It guarantees the uniqueness of the solution to this boundary value problem. Only the outgoing waves are therefore allowed and the energy flux is positive.

2.2 DtN boundary conditions

It is possible to bound the computational domain by imposing a relation between the unknown function and its normal derivative, on an artificial boundary Γ_{art}. This condition denoted by DtN (Dirichlet to Neumann), replaces the radiation condition at infinity. It represents the characteristic impedance outside the computational domain Ω. This domain is limited internally by Γ and externally by Γ_{art}. Denoting by M the DtN operator, the value boundary problem (1) can be reformulated as follows:

$$\begin{cases} \Delta u + k^2 u = \quad 0 \quad \text{in} \quad \Omega \\ u = \quad g \quad \text{on} \quad \Gamma \\ \frac{\partial u}{\partial \boldsymbol{n}} = -Mu \text{ on} \quad \Gamma_{art} \end{cases} . \tag{2}$$

It is known that this boundary value problem admits a unique solution [15]. The exact nonlocal condition introduced by [8], translates the fact that the artificial boundary Γ_{art} does not yield any artificial reflection. It is therefore a Transparent Boundary Condition (TBC) in which Mu can be expressed as an expansion into Hankel functions of the first kind. This Hankel series expansion is obtained from an analytic solution of the Helmholtz equation in the domain outside of Ω. It is limited to an order m such that $kR \leq m$, in order to guarantee uniqueness of the solution to the boundary value problem (2).

Other methods consist of taking only approximations to the TBC. These conditions are then termed 'absorbing' (ABC). Thus, we do not encounter difficulties inherent to the non-locality of the TBC. However, artificial reflections on Γ_{art} are not avoided. The simplest idea is to impose the Robin condition, namely:

$$\frac{\partial u}{\partial \boldsymbol{n}} = -M_0 u = iku \quad \text{on} \quad \Gamma_{art} \tag{3}$$

This condition, which resembles the Sommerfeld radiation condition, does not perform too well. To reduce reflections on the artificial boundary some authors have proposed boundary conditions of higher order. Using theory of pseudo- differential operators Enquist and Majda [5] have developed a sequence of local ABCs of increasing order. Bayliss and Turkel [1] use an asymptotic development in $1/r$ of the solution u in order to form a sequence of local operators

$$\prod_{m=1}^{p} \left(-ik + \frac{\partial}{\partial n} + \frac{4m-3}{2R}\right) u = 0 \quad \text{on} \quad \Gamma_{art} \tag{4}$$

The operator M is then obtained from (4), in practice $p = 2$ is often used. Thanks to the work of Antoine et al. [2], the authors of Ref. [4, 14] have extended the use of the second order Bayliss-Turkel boundary conditions to arbitrarily shaped convex artificial boundaries without loss of practical accuracy. This new condition, designed by Generalised Bayliss-Turkel non reflecting condition, is easy to implement in any finite element based code. It allows smaller computational domains and leads to better computational efficiencies both in 2D and 3D.

2.3 Variational formulation

In order to apply the Finite Element method, the problem is reformulated in the variational form: Find u in $H_1(\Omega)$ such that:

$$a(u, v) = b(v) \ \forall \ v \in \ H_1(\Omega) \tag{5}$$

where a is a symmetric bilinear form defined on $H_1(\Omega) \times H_1(\Omega)$ by

$$a(u, v) = \int_\Omega \nabla u \nabla v d\Omega \ - \ k^2 \int_\Omega uv d\Omega \ + \int_{\Gamma_{art}} Muv d\Gamma \tag{6}$$

and b is a linear form defined on $L_2(\Omega)$ by :

$$b(v) = \int_\Omega fv d\Omega \tag{7}$$

In the Galerkin- Least-Squares discretization scheme the variational formulation is adjusted in order to account for the residual of the partial differential equation. Under this constraint, the weak form of the equation is

$$a_{GLS}(u, v) = b(v), \quad \text{with} \quad a_{GLS}(u, v) = a(u, v) + \tau \int_\Omega \mathcal{H}u \mathcal{H}v d\Omega \tag{8}$$

where $\mathcal{H}$ is the Helmotz differential operator $\mathcal{H}(.) = \Delta(.) + k^2(.)$ and τ is a parameter whose choice depends on some design criterion.

2.4 Analytic and Numerical computation of the parameter τ

For a regular mesh, consisting of bilinear finite elements and with a constant mesh size h, the parameter τ determined by Harrari et al. [6] is expressed as

$$\tau = \frac{1}{k^2} \left(1 - \frac{6(4 - f_x - f_y - 2 f_x f_y)}{(kh)^2 (2 + f_x)(2 + f_y)} \right) \quad \text{with} \quad f_x = \cos(kh \ \cos\theta) , f_y = \cos(kh \ \sin\theta)$$

where θ is the direction of propagation of the plane waves. Two values of τ corresponding to $\theta = 0$ and to $\theta = 22.5$ deg. are often used [6, ?].

For non regular meshes, we propose as an alternative to compute the parameter τ numerically. This is done within each finite element. The basic idea starts from the system of equations which arise from the discretized version of the variational formulation above:

$$[K]_e \{u\}_e - \tau k^2 [M]_e \{u\}_e = \{0\}$$

where $[K]_e$ and $[M]_e$ are the element stiffness and mass matrices respectively. Multiplying both sides of the above expression by the row vector $\{u\}_e^\star = \{\bar{u}\}_e^T$, the transpose and complex conjugate of u, yields the desired expression for the parameter τ:

$$\tau = \frac{\{u\}_e^\star [K]_e \{u\}_e}{k^2 \{u\}_e^\star [M]_e \{u\}_e} \quad \text{with} \quad \{u\}_e = \{exp\left(i(k\cos(\theta)x_i + k\sin(\theta)y_i)\right)\}$$

and x_i, y_i are the coordinates of the nodes of the element. It is straightforward to extend this approach to the three-dimensional case and to isoparametric finite elements.

3 Solution method

In what follows we use piecewise bilinear polynomial functions over quad elements. The discretized version of (6) or (8) result in a linear system of equations of the form

$$Au = b \tag{9}$$

The coefficient matrix A in the above system, is sparse, symmetric, and complex. Note that it is also non-Hermitian and that it is not diagonally dominant.

As can be seen, the effect of the DtN condition in the discretization scheme by standard finite elements results only in the addition of the complex matrix C to form the global matrix A. In addition, the terms C_{ij} are zero whenever the nodes with index i or j do not belong to the artificial boundary Γ_{art}. If the DtN condition is non-local then clearly the terms C_{ij} are nonzero for all nodes of index i, or j belonging to Γ_{art}, and this can destroy the band structure of the matrix if a particular numbering is not used.

We exclude the use of direct methods in this study because of their potential excessive cost – especially for 3D problems. Among iterative methods, preconditioned Krylov subspace techniques [12] are the most general-purpose and appear to a good alternative to direct solvers.

A preconditioned Krylov subspace method for solving the linear system (9) consists of an accelerator and a preconditioner [12]. In what follows we call M the preconditioning matrix, so that, for example, the right-preconditioned system

$$AM^{-1}y = b \qquad \text{where} \quad x = M^{-1}y \tag{10}$$

is solved instead of the original system (9). The above system is solved via an "accelerator", a term used to include a number of methods of the Krylov subspace class. Thus, a right-preconditioned Krylov subspace method computes an approximate solution from the affine space

$$x_0 + \mathrm{Span}\{\mathrm{r}_0, \mathrm{AM}^{-1}\mathrm{r}_0, \cdots, \left(\mathrm{AM}^{-1}\right)^{\mathrm{m}-1}\mathrm{r}_0, \} \ ,$$

which verifies certain conditions. For example, the GMRES algorithm [12] requires that the residual $r_m = b - Ax_m$ has a minimal 2-norm.

The most common way to define the preconditioning matrix M is through Incomplete LU factorizations. An ILU factorization is obtained from an approximate Gaussian elimination process. When Gaussian elimination is applied to a sparse matrix A, a large number of nonzero elements may appear in locations originally occupied by zero elements. These fill-ins are often small elements and may be dropped to obtain Incomplete LU factorizations.

The simplest of these procedures, ILU(0) is obtained by performing the standard LU factorization of A and dropping all fill-in elements that are generated during the process. Thus, the L and U factors have the same pattern as the lower and upper triangular parts of A (respectively). More accurate factorizations denoted by ILU(k) have been defined which drop fill-ins according to their 'levels' in the elimination process, where the levels attempt to reflect size and are defined recursively [12]. We will not consider levels other than level zero in this work.

Another class of preconditioners is based on dropping fill-ins according to their numerical values. One of these methods is ILUT (ILU with Threshold). This procedure uses basically a form of Gaussian elimination which generates the rows of L and U one by one. Small values are dropped during the elimination, using a parameter τ. A second parameter, p, is then used to keep the largest p entries in each of the rows of L and U. This procedure is denoted by $ILUT(\tau, lfil)$ of A. Practical values of $lfil$ and τ are $lfil = \alpha NNZ/n$, with NNZ the number of non-zero entries of the original matrix, n the number of unknowns, α is a positive integer, and $10^{-9} \leq \tau \leq 10^{-3}$. The quotient NNZ/n refers to the average number of nonzero entries in A.

4 Numerical Experiments

Problem 1. The test case we study is the following

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } & \Omega_e \\ \frac{\partial u}{\partial n} + iku = g \text{ in } & \Gamma_{art} \end{cases} \tag{11}$$

where Ω is the unit square $\Omega = (0,1) \times (0,1)$, the function g is selected in such a way that the exact solution has the expression $u(x,y) = exp[ik\cos(\theta)x + k\sin(\theta)y]$. In all cases the numerical solution is compared with this exact solution.

The set of meshes used were structured meshes. The Krylov subspace dimension is set to 20 and the incident angle to $\theta = 45$ degrees in all tests. The initial guess is zero and the GMRES iteration is stopped when the initial residual norm is reduced by a factor of 10^{-8} or when the number of steps exceeds 500. All floating point operations are performed in double precision on a PC intel X86 family, with 512 KB of RAM.

Table 1 shows one set of results, and Figure 1 shows a comparison of the convergence history of ILUT and ILU0 preconditioners. This test was generated with a mesh size of $h = 1/200$. The matrix has $nnz = 361,201$ nonzero elements. The wavelength is $\lambda = 0.628$, so the ratio h/λ is about 314.

Problem 2. In the following set of experiments, we consider the following physical setting. The obstable is a disk of radius $r_0 = 0.5m$. The incident wave is plane with a wavelength λ, and it propagates along the x-axis. The second order Bayliss-Turkel boundary conditions are used on the artificial boundary, located at a distance $2r_0$ from the obstacle, The discretization uses isoparametric elements with 4 nodes. The analytic solution is known and can be found in [15].

$\alpha =$	Drop	# steps to	Final res.	Time	L2-norm of error in %	
$lfil * n/nnz$	tolerance	converge	norm	(sec)	Re. part	Im. part
11,11	1.00E-05	20	1.60E-08	182	0.10	0.11
5,55	1.00E-05	176	2.50E-08	265	0.10	0.11
2,22	1.00E-05	500	7.80E-08	377	5.24	5.34
11,11	1.00E-04	380	2.60E-08	628	0.10	0.11
5,55	1.00E-04	350	2.76E-08	492	0.10	0.11
2,22	1.00E-04	500	5.06E-04	363	3.00	3.30
11,11	1.00E-03	500	5.10E-04	402	5.92	6.36
5,55	1.00E-03	500	5.70E-04	610	5.91	6.36
2,22	1.00E-03	500	1.33E-03	344	6.90	7.60

Table 1. Numerical results with ILUT-GMRES aith various parameters.

4.1 Impact of discretization

Under the same conditions ($k = 2\pi$) and for different meshes, the accuracy of the results obtained with the GLS scheme are superior to those with the classical Galerkin scheme. The number of iterations and execution times are in both cases almost identical. The least-squares Galerkin scheme leads to a good accuracy even for a rough solution using 10 points per wavelength. The accuracy of the results is comparable with that obtained in [15], see table 2, using a classical Galerkin scheme with an exact non-local Dirichlet to Neumann (DtN) scheme. As a comparison, the errors on the real part obtained by Zebic were $5.15, 1.33, 0.34$ for $\lambda/h = 10, 20, 40$ respectively. The preconditioner selected by Zebic

[15] is based on an incomplete CROUT factorization. In this reference, the finite elements used are triangles with 3 nodes.

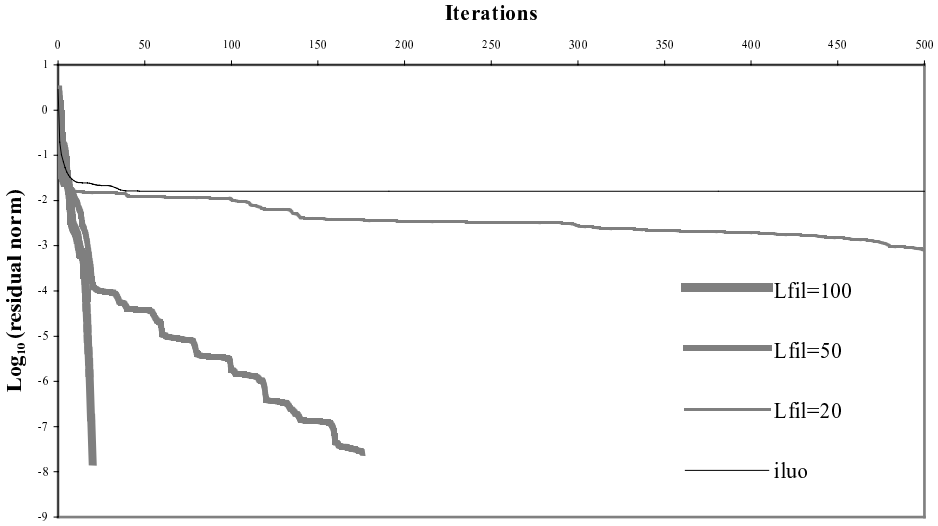

Figure 1 *Convergence histories - residual norms vs iteration*

λ/h	Relative L_2 error on the real part	Relative L_2 error on the imaginary part	Number of Iterations	Time (sec.)
10	4.42 (2.00)	4.51(2.07)	76 (77)	1.37 (1.43)
20	1.14 (0.51)	1.16 (0.50)	138 (139)	4.42 (4.52)
40	0.30 (0.12)	0.31 (0.10)	220 (219)	12.93 (12.83)

Table 2. Error of the finite element solution (Galerkin and Galerkin/Least-squares) for different mesh-sizes. Order 2 Bayliss-Turkel boundary conditions used.

These results show that the Bayliss-Turkel condition gives excellent results for an artificial boundary that is close to the obstacle (here at a distance λ) even with an inaccurate mesh resolution corresponding to $\lambda/h = 10$.

4.2 Impact of the the $lfil$ parameter in the ILUT factorization

This experiment uses the same physical setting with the wave number $k = 2\pi$. The restart value for GMRES (Krylov subspace dimension) is fixed to $m = 10$ and the stopping criterion is to reduce the residual norm by 8 orders of magnitude. ILUT has a second parameter which is a drop tolerance. This is set to 10^{-5}. The mesh size corresponds to $\lambda/h = 40$.

As the results in Table 3 show, the number of iterations required by GMRES to reach convergence decreases when $lfil$ increases in the intervals $[5, 15]$ and $[19, 30]$. However, for $lfil$ in the interval $[15, 19]$, the number of iterations increases with increasing $lfil$, showing that the number of iterations is not necessarily a monotinically decreasing function of $lfil$. This is likely due to the fact that in the case of highly indefinite matrices, the factors L and U may become unstable (i.e., $(LU)^{-1}$ can become huge) – and so the gains in accuracy made in the factorization are overweighed by the instability. For the experiments performed here, the relative L_2 errors of the real parts and imaginary parts (respectively)

are 0.30 % (0.12 % for GLS scheme) and 0.31 %(0.10 % for GLS scheme), respectively. It is interesting also to observe that the number of iterations to reach convergence can be reduced in the interval [15, 19] by changing the drop-tolerance to 10^{-3} (from10^{-5}).

Lfil	5	10	15	18	19	20	21	22	25	30
Iter	220	90	69	78	170	132	75	52	26	14
	(219)	(90)	(69)	(78)	(143)	(139)	(72)	(52)	(26)	(14)
Time	12.93	8.16	8.21	10.06	19.29	16.50	10.71	8.67	6.38	5.53
(sec)	(12.83)	(8.08)	(8.12)	(9.94)	(16.48)	(17.06)	(10.40)	(8.73)	(6.34)	(5.99)

Table 3. Iteration count (Iter) and CPU time (Time) versus the $lfil$ parameter of ILUT. Tolerance is set to 1.e-05. GLS scheme results between parentheses.

In summary, an increase of $lfil$ is not always beneficial. This known fact [12, 9] is well illustrated in the example tested here.

4.3 Performance of GMRES-ILUT for different frequency regimes

We continue with the same physical setting, but in this experiment the fequency λ is now varied. The parameter τ of the GLS method is evaluated numerically in each element. Different mesh sizes are considered depending on the wavenumber. The parameter $lfil$ is selected to be relatively small in order to reduce the cost of computing the ILUT factorization.

Discr. Scheme	Number of nodes	λ/h	Wave # k	Iter. count	% error norm (Re part)	% error norm (Im part)	Sol. time
Galerkin	7421	40	2π	1545	0.30	0.31	64.77
GLS				1541	0.13	0.11	51.66
Galerkin	7421	10	8π	725	16.07	15.85	30.34
GLS				756	8.29	8.33	31.52
Galerkin	29141	40	8π	1490	1.41	1.42	297.33
GLS				1493	0.22	0.20	300.35
Galerkin	29141	20	16π	9770	10.91	10.88	1945.54
GLS				10207	1.54	1.55	2030.96
Galerkin	58101	40	16π	GMRES			
GLS				Stag.			
Galerkin	65160	45	16π	GMRES			
GLS				Stag.			

Table 4. Behavior of ILUT-preconditioned GMRES for different wavenumbers and different mesh sizes

The results reported in Table 4 are obtained with $lfil = 2$. The following observations can be made.

- Consider the situation of small values of the wave number k. In this case the resolution of the mesh corresponding to $\lambda/h = 40$ (40 mesh points per per wavelength), yields an

Discr. Scheme	Number of nodes	λ/h	Wave # k	Iter. count	% error norm (Re part)	% error norm (Im part)	Sol. time
Galerkin	58101	40	16π	8387	6.03	6.03	3113.58
GLS				8169	3.82	3.82	2957.46
Galerkin	65160	45	16π	13304	5.77	5.77	5609.76
GLS				13140	4.04	4.05	5228.25

Table 5. Behavior of ILUT-preconditioned GMRES for different wavenumbers and different mesh sizes. Case $lfil = 1$

excellent agreement between the results of the finite element method and the analytical solution. The two discretization schemes give essentially the same accuracy and require almost the exact same number of GMRES steps to converge.

- Consider the case when the wavenumber is a little larger ($k = 8\pi$). In this case, when the mesh resolution is low ($\lambda/h = 10$), only the Galerkin-Least-squares approach leads to a good agreement between the resulting solution and the analytic solution. For a higher resolution ($\lambda/h = 40$), both discretization schemes yield an excellent agreement between the results of the finite element method and the analytical solution. The solution cost is almost identical both schemes. Note also that a relatively small value of the Krylov subspace dimension suffices to yield convergence.
- We consider the situation of a relatively larger value of the wavenumber ($k = 16\pi$). When a mesh resolution of 20 points per wavelength, the accuracy of the result obtained with the GLS scheme is excellent in contrast with the one obtained with the classical Galerkin method. When the mesh resolution doubles ($\lambda/h = 40$), then ILUT-GMRES stagnates. As mentioned above, this is likely due to an unstable factorization. To check this we considered a less accurate ILUT factorization which uses a smaller value of $lfil$. The results are shown in Table 5.

Increasing the mesh resolution leads to more accurate numerical results for the Galerkin scheme in spite of the fact that the Babuska condition ($k^3h^2 < 1$)[7] is not satisfied in the case when $\lambda/h = 40$ ($k^3h^2 = 1.24$ in this situation). Recall that the Babuska condition is only a sufficient condition. Increasing the mesh resolution to 45 points per wavelength, which gives $k^3h^2 = 0.97$, yields also a slight improvement in accuracy. In contrast, the GLS scheme does not lead in this case to a reduction of the accuracy despite the fact that the Babuska condition is satisfied. This may be attributed to the choice of the parameter τ which is tightly linked to the scheme. This merits further study. In any case, the accuracy obtained by the GLS scheme are superior to those of the classical Galerkin scheme.

5 Conclusion

The performance of the complex version of ILUT was good for low, medium, and high frequencies, provided appropriate fill-in levels and drop parameters are used to compute the ILU factorization.

The influence of the mesh resolution, the wave number value and the fill-in parameter on the accuracy and the computational performance has been analyzed, in the second test problem, which is a more realistic acoustic benchmark. The GLS scheme is more accurate than the classical Galerkin scheme for low mesh resolutions. A new formula for the GLS stabilization method was also proposed. The second order Bayliss-Turkel boundary condition performs well compared with the exact DtN. The optimal value of the fill-in parameter to use depends on the wave number. A somewhat surprising finding is that for high wave numbers the fill-in parameter must sometimes be reduced to obtain a useable incomplete factorization. Indeed, since the matrix may be highly indefinite, an accurate ILUT factorization often yields unstable factors.

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