Bits, Bytes, and Integers

CSci 2021: Machine Architecture and Organization
Lectures #2-4, January 23rd-28th, 2015

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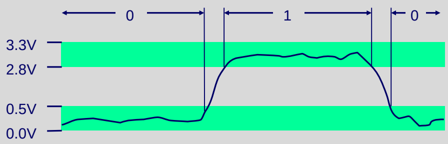
Binary Representations

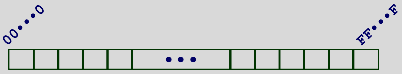


Encoding Byte Values

- **Byte = 8 bits**
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters '0' to '9' and 'A' to 'F'
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Byte-Oriented Memory Organization



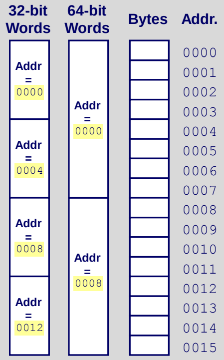
- **Programs Refer to Virtual Addresses**
  - Conceptually very large array of bytes
  - Actually implemented with hierarchy of different memory types
  - System provides address space private to particular "process"
    - Program being executed
    - Program can clobber its own data, but not that of others
- **Compiler + Run-Time System Control Allocation**
  - Where different program objects should be stored
  - All allocation within single virtual address space

Machine Words

- **Machine Has "Word Size"**
  - Nominal size of integer-valued data
    - Including addresses
  - Most current machines use 32 bits (4 bytes) words
    - Limits addresses to 4GB
    - Becoming too small for memory-intensive applications
  - High-end systems use 64 bits (8 bytes) words
    - Potential address space ≈ 1.8 X 10^{19} bytes
    - x86-64 machines support 48-bit addresses: 256 Terabytes
  - Machines support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Data Representations

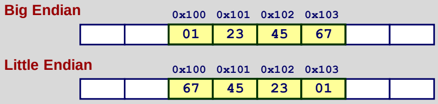
C Data Type	Typical 32-bit	Intel IA32	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	4	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	8	10/12	10/16
pointer	4	4	8

Byte Ordering

- **How should bytes within a multi-byte word be ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, Internet convention
    - Least significant byte has highest address
  - Little Endian: x86, VAX
    - Least significant byte has lowest address

Byte Ordering Example

- **Big Endian**
  - Least significant byte has highest address
- **Little Endian**
  - Least significant byte has lowest address
- **Example**
  - Variable x has 4-byte representation 0x01234567
  - Address given by &x is 0x100



Reading Byte-Reversed Listings

- **Disassembly**
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- **Example Fragment**

```
Address      Instruction Code          Assembly Rendition
8048365:     5b                        pop     %ebx
8048366:     81 c3 ab 12 00 00         add     $0x12ab,%ebx
804836c:     83 bb 28 00 00 00 00      cmpl    $0x0,0x28(%ebx)
```

- **Deciphering Numbers**
  - Value: 0x12ab
  - Pad to 32 bits: 0x000012ab
  - Split into bytes: 00 00 12 ab
  - Reverse: ab 12 00 00

Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * creates byte array

```
void show_bytes(unsigned char *start, int len){
  int i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:
%p: Print pointer
%x: Print Hexadecimal

`show_bytes` Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((unsigned char *) &a, sizeof(int));
```

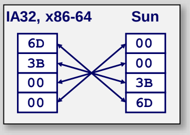
Result (Linux):

```
int a = 15213;
0xbffffcb8  0x6d
0xbffffcb9  0x3b
0xbffffcba  0x00
0xbffffcbb  0x00
```

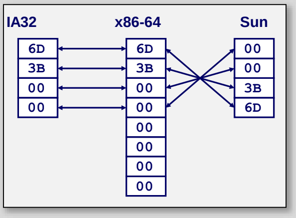
Representing Integers

Decimal:	`15213`
Binary:	`0011 1011 0110 1101`
Hex:	`3 B 6 D`

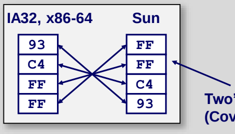
int A = 15213;



long int C = 15213;



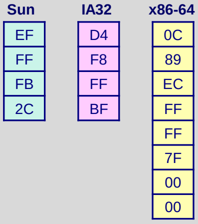
int B = -15213;



Two's complement representation (Covered later)

Representing Pointers

```
int B = -15213;
int *P = &B;
```



Different compilers & machines assign different locations to objects

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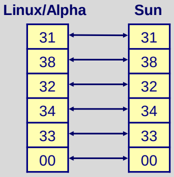
Representing Strings

```
char S[6] = "18243";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character "0" has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



Aside: ASCII table

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
0x0_	\0	^A	^B	^C	^D	^E	^F	^G	^H	\t	\n	^K	^L	^M	^N	^O
0x1_	^P	^Q	^R	^S	^T	^U	^V	^W	^X	^Y	^Z	ESC	FS	GS	RS	US
0x2_	SPC	!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
0x3_	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
0x4_	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0x5_	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
0x6_	`	a	b	c	d	e	f	g	h	I	j	k	l	m	n	o
0x7_	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

Today: Bits, Bytes, and Integers

- Representing information as bits
- **(Logistics interlude)**
- **Bit-level manipulations**
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
- Summary

Homework turn-in process

- **For full credit: turn in at the beginning of class on the due date**
  - On-time = 3:35pm, or when I start lecturing, whichever is later
  - Yes, this means you have to come to class on time (that day)
- **We strongly recommend typing your assignments on a computer, not hand-writing**
- ***Late* submissions only will be online using the Moodle**
- **Do not turn in paper assignments at other times**
  - This helps us stay organized

2021-dedicated VMs now available

- **SSH into: x*A*-*B*.cselabs.umn.edu**
  - Where A is 21, 22, or 23
  - And B is 01, 02, 03, 04, or 05
  - E.g., x22-02.cselabs.umn.edu
- **32-bit version of Ubuntu Linux version 14.04**
- **Do not run graphical programs (Firefox, etc.) on these machines (it would be slow anyway)**
- **If you prefer to use other CSE Labs Linux machines, give the `-m32` option to GCC to get 32-bit binaries**

Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode “True” as 1 and “False” as 0

And (math: ⋀)

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or (math: ⋁)

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not (math: ¬)

- **~A = 1 when A=0**

~	
0	1
1	0

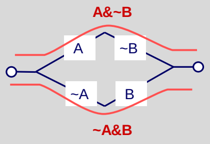
Exclusive-Or “xor” (math: ⊕)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

Application of Boolean Algebra

- **Applied to Digital Systems by Claude Shannon**
  - 1937 MIT Master’s Thesis
  - Reason about networks of relay switches
    - Encode closed switch as 1, open switch as 0



Connection when

A&~B | ~A&B

= A^B

General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- **All of the Properties of Boolean Algebra Apply**

Representing & Manipulating Sets

- **Representation**
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$
    - 01101001 { 0, 3, 5, 6 }
    - *76543210*
    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- **Operations**

&	Intersection	01000001	{ 0, 6 }
|	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	Symmetric difference	00111100	{ 2, 3, 4, 5 }
~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
  - Apply to any “integral” data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (Char data type)**
  - ~0x41 → 0xBE
    - ~01000001_2 → 10111110_2
  - ~0x00 → 0xFF
    - ~00000000_2 → 11111111_2
  - 0x69 & 0x55 → 0x41
    - 01101001_2 & 01010101_2 → 01000001_2
  - 0x69 | 0x55 → 0x7D
    - 01101001_2 | 01010101_2 → 01111101_2

Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - &&, ||, !
    - View 0 as “False”
    - Anything nonzero as “True”
    - Always return 0 or 1
    - Early termination (AKA “short-circuit evaluation”)
- **Examples (char data type)**
  - !0x41 → 0x00
  - !0x00 → 0x01
  - !!0x41 → 0x01

  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Shift Operations

- **Left Shift: x << y**
  - Shift bit-vector x left y positions
    - – Throw away extra bits on left
    - Fill with 0’s on right
- **Right Shift: x >> y**
  - Shift bit-vector x right y positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0’s on left
  - Arithmetic shift
    - Replicate most significant bit on right
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

Argument `x`	`01100010`
`<< 3`	`00010`*`000`*
Log. `>> 2`	*`00`*`011000`
Arith. `>> 2`	*`00`*`011000`

Argument `x`	`10100010`
`<< 3`	`00010`*`000`*
Log. `>> 2`	*`00`*`101000`
Arith. `>> 2`	*`11`*`101000`

Exercise break: flip case

```
/* Convert lowercase to uppercase and vice-versa,
   return any other characters unchanged */
char flip_case(char c){
    if (c >= 'A' && c <= 'Z') {
        /* 0x41 through 0x5A */
       return  c | 0x20 ;
    } else if (c >= 'a' && c <= 'z') {
       /* 0x61 through 0x7A */
       return  c & ~0x20;
    } else {
        return c;
    }
}
```

- **Fill in the blanks, using bitwise operators**

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - **Representation: unsigned and signed**
  - Conversion, casting
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- Summary

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two’s Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`y`	-15213	`C4 93`	`11000100 10010011`

- **Sign Bit**
  - For 2’s complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1
- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
`UMax`	65535	`FF FF`	`11111111 11111111`
`TMax`	32767	`7F FF`	`01111111 11111111`
`TMin`	-32768	`80 00`	`10000000 00000000`
`-1`	-1	`FF FF`	`11111111 11111111`
`0`	0	`00 00`	`00000000 00000000`

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - |*TMin* | = *TMax* + 1
    - Asymmetric range
  - *UMax* = 2 * *TMax* + 1
- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **⇒ Can Invert Mappings**
  - U2B(*x*) = $B2U^{-1}(x)$
    - Bit pattern for unsigned integer
  - T2B(*x*) = $B2T^{-1}(x)$
    - Bit pattern for two's comp integer

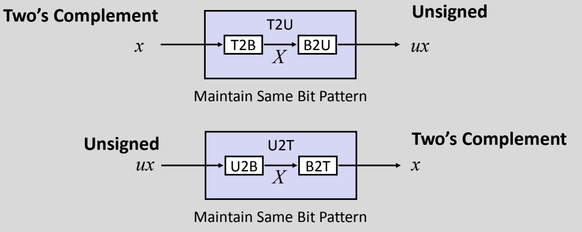
Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - **Conversion, casting**
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
- Summary

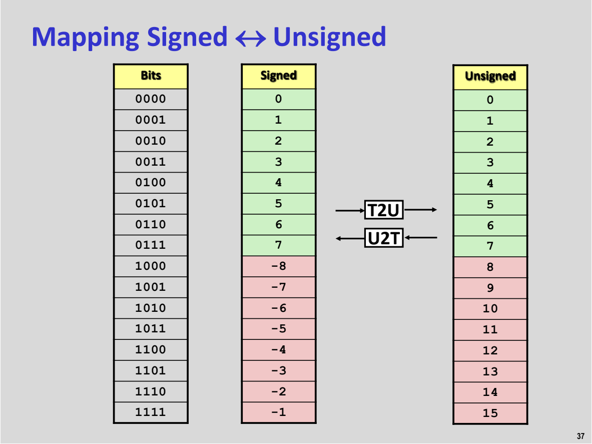
Announcement interlude: Lab 1 out

- **Lab 0 (Hello, world) is due tonight**
- **Lab 1 on data representation is out**
- **Basic idea: puzzles implementing operations with other operations**
  - E.g., implement logical right shift using only arithmetic right shift
- **Most problems relate to bitwise operations and two's complement rules**
  - I.e., you can start working on them now
- **Increasing difficulty, try the easier ones first**
- **Two questions relating to floating point**

Mapping Between Signed & Unsigned

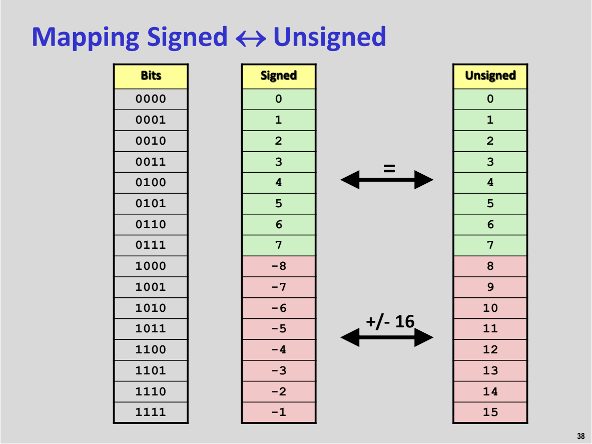


- **Mappings between unsigned and two's complement numbers: keep bit representations and reinterpret**



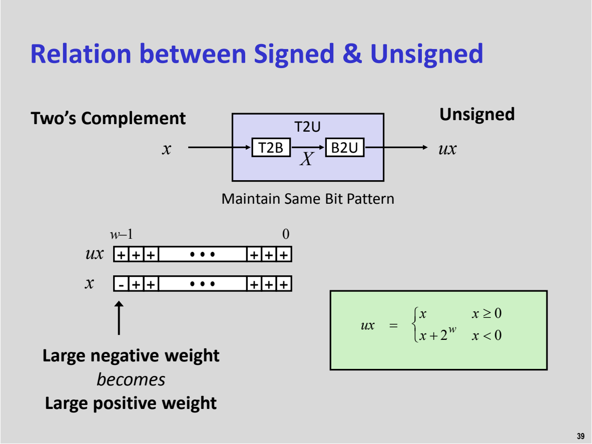
Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15



Mapping Signed ↔ Unsigned

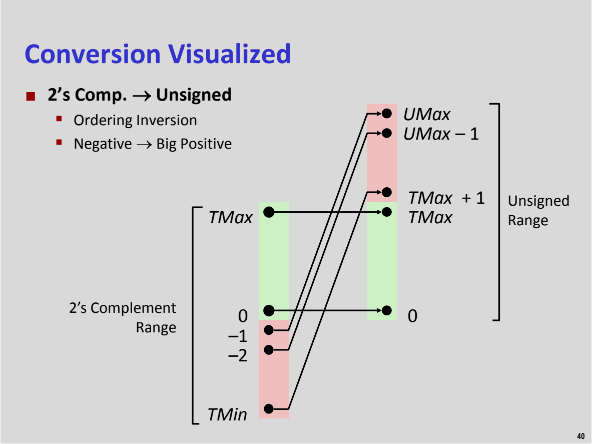
Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15



Relation between Signed & Unsigned

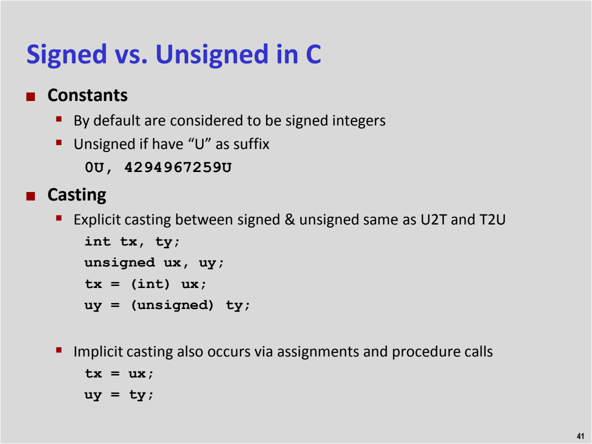
Large negative weight
becomes
Large positive weight

$$ux = \begin{cases} x & x \ge 0 \\ x + 2^w & x < 0 \end{cases}$$



Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

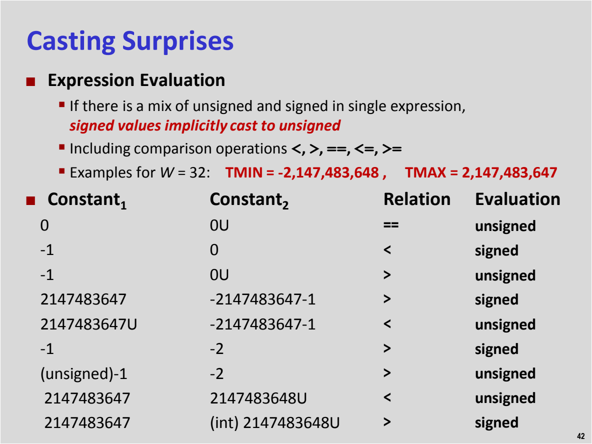
```
0U, 4294967259U
```

- **Casting**
  - Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```



Casting Surprises

- **Expression Evaluation**
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations <, >, ==, <=, >=
  - Examples for *W* = 32: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

- **Similar to code found in FreeBSD's implementation of getpeername**
- **There are legions of smart people trying to find vulnerabilities in programs**

Typical Usage

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf(“%s\n”, mybuf);
}
```

Malicious Usage

```
/* Declaration of library function memcpy */
void *memcpy(void *dest, void *src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

Summary
Casting Signed ↔ Unsigned: Basic Rules

- **Bit pattern is maintained**
- **But reinterpreted**
- **Can have unexpected effects: adding or subtracting 2^w**

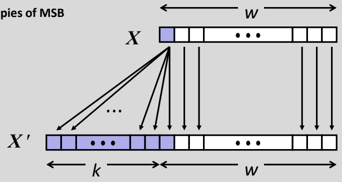
- **Expression containing signed and unsigned int**
  - `int` is cast to `unsigned`!!

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - **Expanding, truncating**
  - Addition, negation, multiplication, shifting
- Summary

Sign Extension

- **Task:**
  - Given *w*-bit signed integer *x*
  - Convert it to *w*+*k*-bit integer with same value
- **Rule:**
  - Make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1},\ldots,x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2},\ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**

Summary:
Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
  - Unsigned: zeros added (“zero extension”)
  - Signed: sign extension
  - Both yield expected result
- **Truncating (e.g., unsigned to unsigned short)**
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behaviour

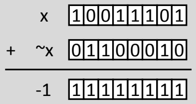
Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - **Addition, negation, multiplication, shifting**
- Summary

Negation: Complement & Increment

- **Claim: Following Holds for 2’s Complement**

  `~x + 1 == -x`
- **Complement**
  - Observation: `~x + x == 1111…111 == -1`



- **Where would we fill in gaps for a more complete proof?**
- **Note: operation can apply to unsigned as well**
- **Two values for which *x* and *-x* have the same sign**

Complement & Increment Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

x = 0

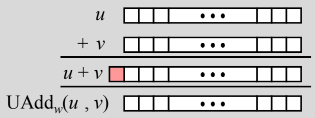
	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

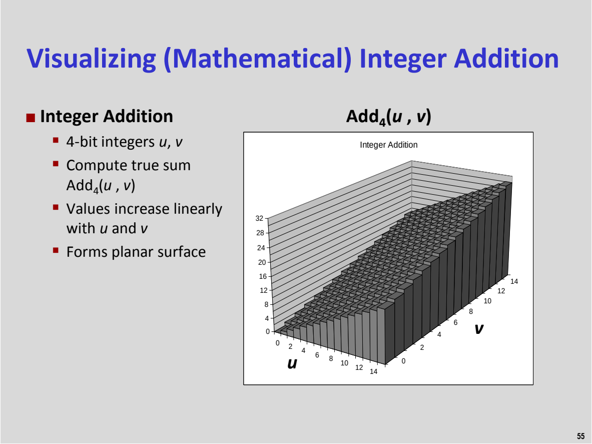
Discard Carry: *w* bits



- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

  $s \;=\; \mathrm{UAdd}_w(u\,,v) \;=\; u + v \bmod 2^w$

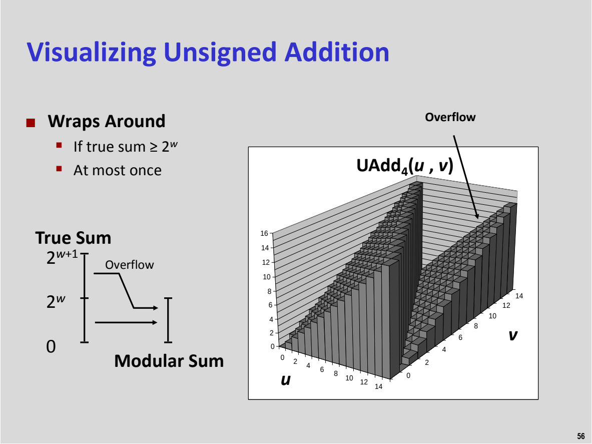
$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$



Visualizing (Mathematical) Integer Addition

- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u, v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

$\text{Add}_4(u, v)$



Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ 2^w
  - At most once

Mathematical Properties

- **Modular Addition Forms an *Abelian Group***
  - **Closed** under addition

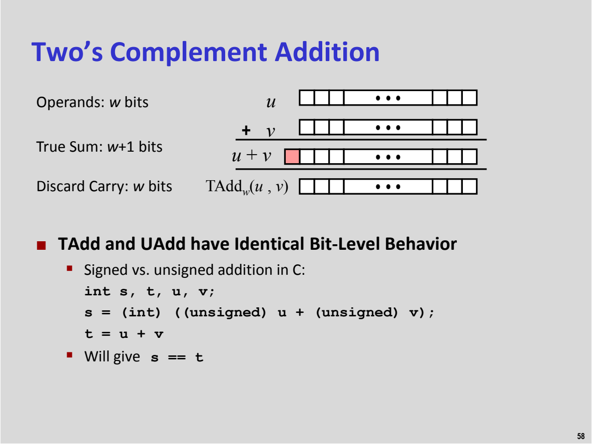
    $0 \le \text{UAdd}_w(u, v) \le 2^w - 1$
  - **Commutative**

    $\text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)$
  - **Associative**

    $\text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$
  - **0** is additive identity

    $\text{UAdd}_w(u, 0) = u$
  - Every element has additive **inverse**
    - Let $\text{UComp}_w(u) = 2^w - u$

      $\text{UAdd}_w(u, \text{UComp}_w(u)) = 0$



Two's Complement Addition

Operands: *w* bits

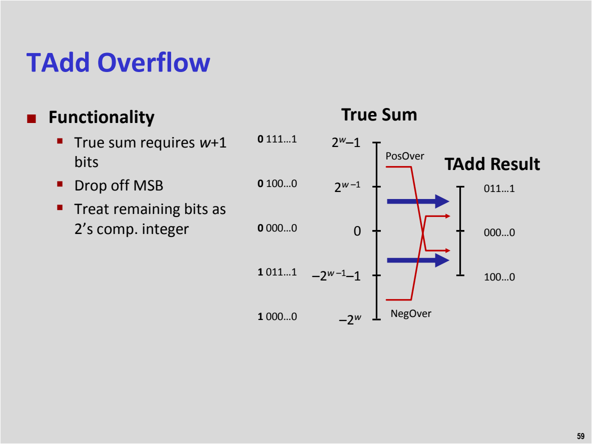
True Sum: *w*+1 bits

Discard Carry: *w* bits

- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

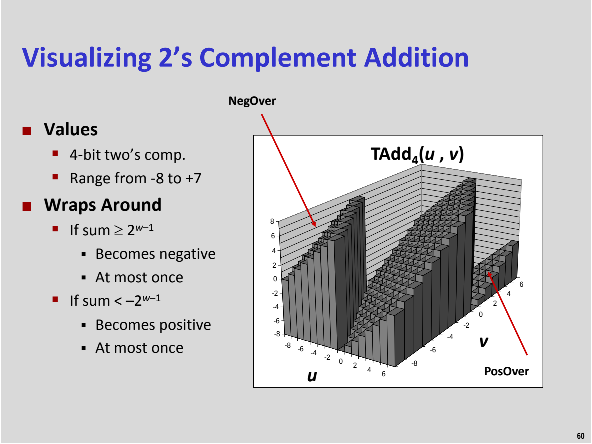
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`



TAdd Overflow

- **Functionality**
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer

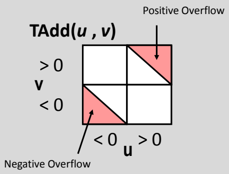


Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum ≥ 2^{w-1}
    - Becomes negative
    - At most once
  - If sum < -2^{w-1}
    - Becomes positive
    - At most once

Characterizing TAdd

- **Functionality**
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

Mathematical Properties of TAdd

- **Isomorphic Group to unsigneds with UAdd**
  - $\text{TAdd}_w(u, v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
    - Since both have identical bit patterns
- **Two's Complement Under TAdd Forms a Group**
  - Closed, Commutative, Associative, 0 is additive identity
  - Every element has additive inverse

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Exercise break: ten's complement

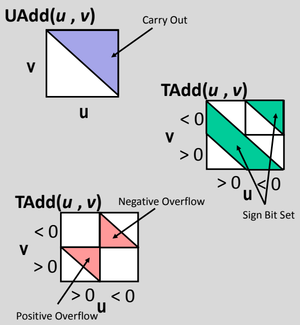
- **Before digital computers, there were mechanical computers that used base 10**
- **There's an analog of two's complement called ten's complement that works in decimal**
- **Suppose we have an adding machine with 10 decimal digits, 10^4 instead of 2^{32}.**
- **What should be the ten's complement representation of -21?**
- **I.e., we want a number *x* so that adding *x* is the same as subtracting 21, when you only have 4 digits**

Ten's complement answer

- **We want x ≡ -21 mod 10000, or x + 21 + 10000k = 0 for integer k**
- **x = 10000 - 21 = 9979**
- **(The equivalent of ~ is called nines' complement: ~21 = 9978)**

Signed/Unsigned Overflow Differences

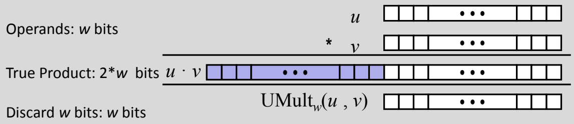
- **Unsigned:**
  - Overflow if carry out of last position
  - Also just called "carry" (C)
- **Signed:**
  - Result wrong if input signs are the same but output sign is different
  - In CPUs, unqualified "overflow" usually means signed (O or V)



Multiplication

- **Computing Exact Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **Ranges**
  - Unsigned: $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
    - Up to 2*w* bits
  - Two's complement min: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
    - Up to 2*w*–1 bits
  - Two's complement max: $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
    - Up to 2*w* bits, but only for $(TMin_w)^2$
- **Maintaining Exact Results**
  - Would need to keep expanding word size with each product computed
  - Done in software by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C



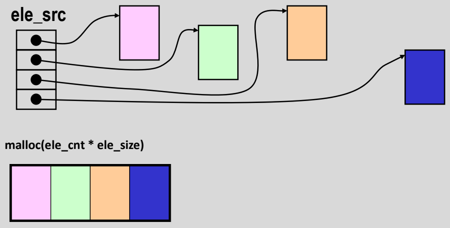
- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

  UMult$_w$(u , v)= $u \cdot v$ mod 2^w

Code Security Example #2

- **SUN XDR library**
  - Widely used library for transferring data between machines

```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size);
```



XDR Code

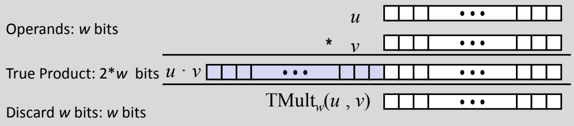
```
void* copy_elements(void *ele_src[], int ele_cnt, size_t ele_size) {
    /*
     * Allocate buffer for ele_cnt objects, each of ele_size bytes
     * and copy from locations designated by ele_src
     */
    void *result = malloc(ele_cnt * ele_size);
    if (result == NULL)
       /* malloc failed */
       return NULL;
    void *next = result;
    int i;
    for (i = 0; i < ele_cnt; i++) {
        /* Copy object i to destination */
        memcpy(next, ele_src[i], ele_size);
       /* Move pointer to next memory region */
       next += ele_size;
    }
    return result;
}
```

XDR Vulnerability

malloc(ele_cnt * ele_size)

- **What if:**
  - `ele_cnt` = 2^{20} + 1
  - `ele_size` = 4096 = 2^{12}
  - Allocation = ??
- **How can I make this function secure?**

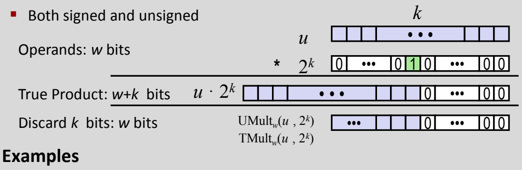
Signed Multiplication in C



- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` gives `u * 2`^{k}
  - Both signed and unsigned



- **Examples**
  - `u << 3          ==  u * 8`
  - `u << 5 - u << 3    ==    u * 24`
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Compiled Multiplication Code

C Function

```
int mul12(int x)
{
  return x*12;
}
```

Compiled Arithmetic Operations

```
leal (%eax,%eax,2), %eax
sall $2, %eax
```

Explanation

```
t <- x+x*2
return t << 2;
```

- **C compiler automatically generates shift/add code when multiplying by constant**

Background: Rounding in Math

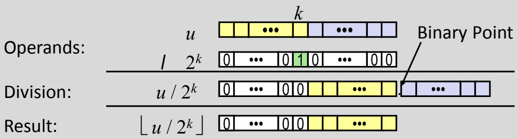
- **How to round to the nearest integer?**
- **Cannot have both:**
  - round(x + k) = round(x) + k (k integer), “translation invariance”
  - round(-x) = -round(x) “negation invariance”
- **$\lfloor \mathtt{x} \rfloor$, read “floor”: always round down (to -∞):**
  - **$\lfloor 2.0 \rfloor = 2$, $\lfloor 1.7 \rfloor = 1$, $\lfloor -2.2 \rfloor = -3$**
- **$\lceil \mathtt{x} \rceil$, read “ceiling”: always round up (to +∞):**
  - **$\lceil 2.0 \rceil = 2$, $\lceil 1.7 \rceil = 2$, $\lceil -2.2 \rceil = -2$**
- **C integer operators mostly use round to zero, which is like floor for positive and ceiling for negative**

Divison in C

- **Integer division /: rounds towards 0**
  - Choice (settled in C99) is historical, via FORTRAN and most CPUs
- **Division by zero: undefined, usually fatal**
- **Unsigned division: no overflow possible**
- **Signed division: overflow *almost* impossible**
  - Exception: TMin/-1 is un-representable, and so undefined
  - On x86 this too is a default-fatal exception

Unsigned Power-of-2 Divide with Shift

- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor \mathtt{u} / 2^k \rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Compiled Unsigned Division Code

C Function

```
unsigned udiv8(unsigned x)
{
  return x/8;
}
```

Compiled Arithmetic Operations

```
shrl $3, %eax
```

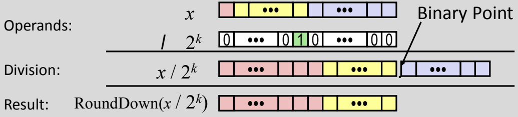
Explanation

```
# Logical shift
return x >> 3;
```

- **Uses logical shift for unsigned**
- **For Java Users**
  - Logical shift written as >>>

Signed Power-of-2 Divide with Shift

- **Quotient of Signed by Power of 2**
  - `x >> k` gives $\lfloor \mathtt{x} / 2^k \rfloor$
  - Uses arithmetic shift
  - Rounds wrong direction when `u < 0`

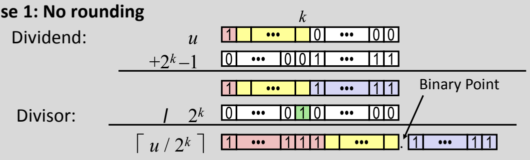


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct Power-of-2 Divide

- **Quotient of Negative Number by Power of 2**
  - Want $\lceil$ `x / 2`^{k} $\rceil$ (Round Toward 0)
  - Compute as $\lfloor$ `(x+2`^{k}`-1)/ 2`^{k} $\rfloor$
    - In C: `(x + (1<<k)-1) >> k`
    - Biases dividend toward 0

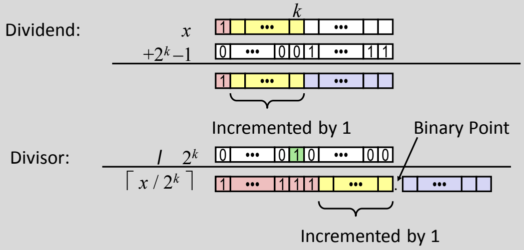
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Compiled Signed Division Code

C Function

```
int idiv8(int x)
{
  return x/8;
}
```

Compiled Arithmetic Operations

```
  testl %eax, %eax
  js   L4
L3:
  sarl $3, %eax
  ret
L4:
  addl $7, %eax
  jmp  L3
```

Explanation

```
  if x < 0
    x += 7;
  # Arithmetic shift
  return x >> 3;
```

- **Uses arithmetic shift for int**
- **For Java Users**
  - Arith. shift written as >>

Remainder operator

- **Written as % in C**
- **`x % y` is the remainder after division `x / y`**
- **E.g., `x % 10` is the lowest digit of non-negative `x`**
- **Behavior for negative values matches /'s rounding toward zero**
  - `b*(a / b) + (a % b) = a`
- **I.e. sign of remainder matches sign of dividend**
- **(Some other languages have other conventions: sign of result equals sign of divisor, sometimes distinguished as "modulo", or always positive)**

Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2w

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Arithmetic: Basic Rules

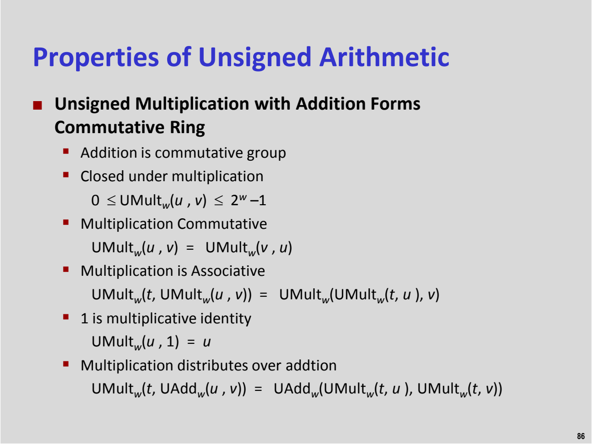
- **Unsigned ints, 2's complement ints are isomorphic rings: isomorphism = casting**

- **Left shift**
  - Unsigned/signed: multiplication by 2^k
  - Always logical shift

- **Right shift**
  - Unsigned: logical shift, div (division + round to zero) by 2^k
  - Signed: arithmetic shift
    - Positive numbers: div (division + round to zero) by 2^k
    - Negative numbers: div (division + round away from zero) by 2^k Use biasing to fix

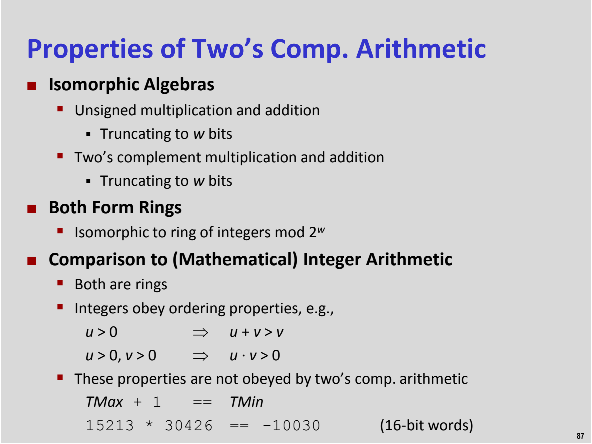
Today: Integers

- Representation: unsigned and signed
- Conversion, casting
- Expanding, truncating
- Addition, negation, multiplication, shifting
- **Summary**



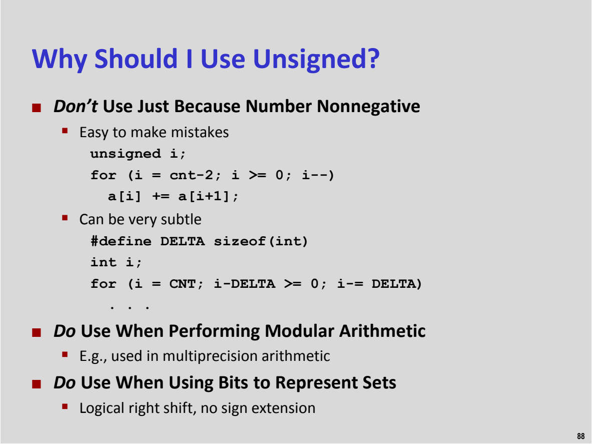
Properties of Unsigned Arithmetic

- **Unsigned Multiplication with Addition Forms Commutative Ring**
  - Addition is commutative group
  - Closed under multiplication
    $0 \leq \mathrm{UMult}_w(u\,,v) \leq 2^w - 1$
  - Multiplication Commutative
    $\mathrm{UMult}_w(u\,,v) = \mathrm{UMult}_w(v\,,u)$
  - Multiplication is Associative
    $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u\,,v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u\,), v)$
  - 1 is multiplicative identity
    $\mathrm{UMult}_w(u\,,1) = u$
  - Multiplication distributes over addtion
    $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u\,,v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u\,), \mathrm{UMult}_w(t, v))$



Properties of Two's Comp. Arithmetic

- **Isomorphic Algebras**
  - Unsigned multiplication and addition
    - Truncating to *w* bits
  - Two's complement multiplication and addition
    - Truncating to *w* bits
- **Both Form Rings**
  - Isomorphic to ring of integers mod 2^w
- **Comparison to (Mathematical) Integer Arithmetic**
  - Both are rings
  - Integers obey ordering properties, e.g.,
    $u > 0 \quad \Rightarrow \quad u + v > v$
    $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
  - These properties are not obeyed by two's comp. arithmetic
    TMax `+ 1 ==` *TMin*
    `15213 * 30426 == -10030` (16-bit words)



Why Should I Use Unsigned?

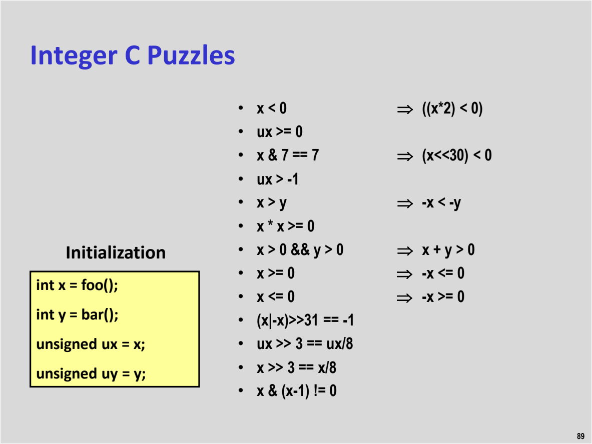
- ***Don't*** **Use Just Because Number Nonnegative**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- ***Do*** **Use When Performing Modular Arithmetic**
  - E.g., used in multiprecision arithmetic
- ***Do*** **Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension



Integer C Puzzles

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- x < 0 ⇒ ((x*2) < 0)
- ux >= 0
- x & 7 == 7 ⇒ (x<<30) < 0
- ux > -1
- x > y ⇒ -x < -y
- x * x >= 0
- x > 0 && y > 0 ⇒ x + y > 0
- x >= 0 ⇒ -x <= 0
- x <= 0 ⇒ -x >= 0
- (x|-x)>>31 == -1
- ux >> 3 == ux/8
- x >> 3 == x/8
- x & (x-1) != 0