Protecting Critical Facets in Layered Manufacturing*

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March 7, 2000

Abstract

In Layered Manufacturing, a three-dimensional polyhedral object is built by slicing its (virtual) CAD model, and manufacturing the slices successively. During this process, support structures are used to prop up overhangs. An important issue is choosing the build direction, as it affects, among other things, the location of support structures on the part, which in turn impacts process speed and part finish. Algorithms are given here that (i) compute a description of all build directions for which a prescribed facet is not in contact with supports, and (ii) compute a description of all build directions for which the total area of all facets that are not in contact with supports is maximum. A simplified version of the first algorithm has been implemented, and test results on models obtained from industry are given.

1 Introduction

Layered Manufacturing (LM) is an emerging technology that is gaining importance in the manufacturing industry. (See e.g. the book by Jacobs [9].)

*This work was funded in part by a joint research grant by DAAD and by NSF.

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This technology makes it possible to rapidly build three-dimensional objects directly from their computer representations on a desktop-sized machine connected to a workstation. A specific process of LM that is widely in use is StereoLithography. The input to this process is the triangulated boundary of a polyhedral CAD model. This model is first sliced by horizontal planes into layers. Then, the object is built layer by layer in the following way. The StereoLithography apparatus consists of a vat of photocurable liquid resin, a platform, and a laser. Initially, the platform is below the surface of the resin at a depth equal to the layer thickness. The laser traces out the contour of the first slice on the surface and then fills the interior, which hardens to a depth equal to the layer thickness. In this way, the first layer is created; it rests on the platform. Then, the platform is lowered by the layer thickness and the just-vacated region is re-coated with resin. The subsequent layers are then built in the same way.

It may happen that the current layer overhangs the previous one. Since this leads to instabilities during the process, so-called *support structures* are generated to prop up the portions of the current layer that overhang the previous layer. (See Figure 1 for an illustration in two dimensions.) These support structures are computed before the process starts. They are also sliced into layers, and built simultaneously with the object. After the object has been built, the supports are removed. Finally, the object is postprocessed in order to remove residual traces of the supports.

An important issue in this process is choosing an orientation of the model so that it can be built in the vertical direction. Equivalently, we can keep the model fixed, and choose a direction in which the model is built layer by layer. This direction is called the *build direction*. It affects the number of layers, the surface finish, the quantity of support structures used, and their location on the object being built.

1.1 Overview of our results

Let $\mathcal{P}$ be the three-dimensional polyhedron that we want to build using LM, let F be a facet of $\mathcal{P}$, and let $\mathbf{d}$ be a build direction. Informally, facet F is not in contact with supports for build direction $\mathbf{d}$, if F can be moved in direction $\mathbf{d}$ without intersecting the polyhedron $\mathcal{P}$. (A formal definition will be given in Section 2.2.)

In this paper, we consider the problem of computing all build directions for which the prescribed facet F is not in contact with supports. This is an important problem because support removal from a facet can affect surface quality and accuracy adversely, thereby impacting the functional properties of critical facets, such as, for instance, facets on gear teeth. This problem, which we describe below, arose from discussions with engineers at Stratasys, Inc.—a Minnesota-based world leader in LM. To our knowledge, the work presented here constitutes the first provably correct, complete, and efficient solution to this important problem; current practice in industry is based on trial and error.

Throughout, we assume that the facets of $\mathcal{P}$ are triangles. (This is the standard STL format used in industry.) The number of facets of $\mathcal{P}$ is denoted by n. We solve the following problems:

Problem 1 *Given a facet F of $\mathcal{P}$, compute a description of all build directions for which F is not in contact with supports.*

Problem 2 *Compute a description of all build directions for which the total area of all facets of $\mathcal{P}$ that are not in contact with supports is maximum.*

In Section 2, we introduce some geometric concepts that are used in the rest of the paper. In particular, we give a formal definition of the notion of a facet being in contact with supports.

In Section 3, we give an algorithm that solves Problem 1 in $O(n^2)$ time. This result also implies an $O(k^2n^2)$-time algorithm that computes all build directions for which any given set of $k \geq 1$ facets of $\mathcal{P}$ is not in contact with supports. We also construct a polyhedron $\mathcal{P}$ that has a facet F such that the set of build directions for which F is not in contact with supports consists of $\Omega(n^2)$ connected components.

We have implemented a simplified version of this algorithm. Test results on models obtained from industry are given in Section 3.7.

In Section 4, we show that by generalizing the algorithm of Section 3, Problem 2 can be solved in $O(n^4)$ time.

The algorithms solving Problems 1 and 2 use fundamental concepts from computational geometry, such as convex hulls, arrangements of line segments, and the overlay of planar graphs. These concepts, however, are applied to points and segments on the unit sphere.

Our algorithm of Section 4 can be used to compute a build direction that approximates the minimum contact-area between the supports and the model. We describe this in Section 5.

1.2 Prior related work

The problem of computing a "good" build direction has been considered in the literature. Asberg et al. [3] give efficient algorithms that decide if a model can be made by StereoLithography without using support structures.

Allen and Dutta [1] consider the problem of minimizing the total area of all parts of the model that are in contact with support structures. They give a heuristic for this problem, but without any analysis of the running time or the quality of the approximation. Bablani and Bagchi [4] present heuristics for improving the accuracy and finish of the part.

In our previous work [11, 12, 13, 18], we have used techniques from computational geometry to compute build directions that optimize various design criteria. In [12], algorithms are given that minimize, for convex polyhedra, the volume of support structures used, and, independently, the total area of those parts of the model that are in contact with support structures. Both algorithms have a running time that is bounded by $O(n^2)$, where n is the number of facets. For general polyhedra, it is shown that a build direction that minimizes the so-called stair-step error can be computed in $O(n \log n)$ time. In [13], algorithms are given that minimize a combination of these measures. (For all measures that involve support structures, the algorithms only work for convex polyhedra.) In [11], algorithms are given that minimize support structures for two-dimensional simple polygons. It is not clear if these algorithms can be implemented such that they run in polynomial time because some complicated functions have to be minimized. (See also Section 5.) Finally, in [18], the implementation of an algorithm that minimizes the number of layers, is discussed. This algorithm works for general polyhedra.

We are not aware of any efficient algorithm that minimizes support structures for general three-dimensional polyhedra.

While writing the final version of the current paper, we became aware of related work by Nurmi and Sack [15]. (Private communication from J.-R. Sack.) They consider the following problem: Given a convex polyhedron A and a set of convex polyhedral obstacles, compute all directions of translations that move A arbitrarily far away such that no collision occurs between A and any of the obstacles. If we take for A a facet F of a polyhedron $\mathcal{P}$, and for the obstacles the other facets of $\mathcal{P}$, then we basically get Problem 1. Our algorithm for solving Problem 1 is similar to that of Nurmi and Sack. However, our algorithm is tailored to our particular application, and takes advantage of the fact that the objects of interest are the facets of a polyhedron. Moreover, we give rigorous proofs that handle all boundary cases—something that is crucial when deploying our algorithm in an actual LM application, since such boundary cases are very common in real-world STL files.

There is also a connection to *aspect graphs*. An aspect graph is a planar graph on the unit-sphere $\mathbb{S}^2$, whose vertices, edges and faces correspond to all different topological views of the polyhedron $\mathcal{P}$. See Bowyer and Dyer [5], and Plantinga and Dyer [16]. To solve Problem 1, we basically have to compute those parts of the aspect graph for which facet F is completely visible. We follow a different, but related, approach: For each facet G of $\mathcal{P}$, $G \neq F$, we compute the set of all build directions for which F is in contact with supports, "because of" facet G. Then the complement of the union of these sets gives the solution to Problem 1. We believe that this approach is more intuitive. Moreover, using this approach, the boundary cases can easier be analyzed. Finally, our approach leads to an algorithm that is easier to implement.

2 Geometric preliminaries

2.1 Polyhedra

In this section, we formally define the class of polyhedra that can be handled by our algorithms. This definition follows Steinitz. (See the book by Steinitz and Rademacher [19], and the paper by Kettner [10].)

Let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be three sets whose elements are called vertices, edges, and facets, respectively, and let $\mathcal{C} := \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$. We call $\mathcal{C}$, together with a symmetric incidence relation on it, a *structural complex*, if

- no two elements of the same set $\mathcal{V}$, $\mathcal{E}$, or $\mathcal{F}$ are incident, and
- for any $v \in \mathcal{V}$, $e \in \mathcal{E}$ and $f \in \mathcal{F}$, if v is incident to e, and e is incident to f, then v is incident to f.

A structural complex is called a *polyhedral complex*, if

- each edge is incident to exactly two vertices,
- each edge is incident to exactly two facets,
- for any vertex v and facet f that are incident, there are exactly two edges that are incident to both v and f,
- each vertex is incident to at least one edge or facet,
- each facet is incident to at least one vertex or edge.

Let v be a vertex of a polyhedral complex. The *neighborhood* of v is defined as the set of all edges and facets that are incident to v. This neighborhood can be partitioned into pairwise disjoint cycles, where each cycle is an alternating sequence of edges and facets. Similarly, we define the *neighborhood* of a facet f

as the set of all vertices and edges that are incident to f. This neighborhood can also be partitioned into disjoint cycles, each one being an alternating sequence of vertices and edges. We assume in this paper that for each facet f, its neighborhood consists of one single cycle.

Assume that for each facet f of a polyhedral complex, the edges on the cycle of f's neighborhood have an orientation. We say that the the polyhedral complex is *orientable*, if for each edge e, the two cycles of the facets that are incident to e have opposite orientations.

A *polyhedron* $\mathcal{P}$ is an orientable polyhedral complex, together with a function that maps the vertices of $\mathcal{V}$ to points in $\mathbb{R}^3$, and the edges of $\mathcal{E}$ to the straight-line segment joining its two endpoints. This function satisfies the following two conditions.

- For each facet f, the function maps the neighborhood of f to the boundary of a simple polygon that is contained in a two-dimensional plane. Hence, we can extend the function such that it maps a facet to the bounded region defined by this planar polygon.

- The images of all vertices, the interiors of all edges and the interiors of all facets are pairwise disjoint.

Our assumption that the neighborhood of each facet f consists of one single cycle translates to the fact that the image of f is a planar simple polygon without holes.

In the rest of this paper, we will not distinguish between vertices, edges, and facets, and their images under the function. An example of a polyhedron is two tetrahedra put together so that they have exactly one vertex or edge in common.

A polyhedron $\mathcal{P}$ has an *interior*, which may consist of several connected components, and an *exterior*. Each facet f has an *inner normal*, which is the unit-length vector that is orthogonal to the plane containing f and that points in the interior of $\mathcal{P}$, and an *outer normal*, which is the unit-length vector that is orthogonal to the plane containing f and that points in the exterior of $\mathcal{P}$.

Throughout this paper, $\mathcal{P}$ denotes a polyhedron in $\mathbb{R}^3$ whose facets are triangles. The number of facets of $\mathcal{P}$ is denoted by n.

We consider each facet of $\mathcal{P}$ as being closed, i.e., the vertices and edges on its boundary belong to the facet. Similarly, an edge of $\mathcal{P}$ is closed in the sense that its two endpoints belong to the edge. Note that no vertex is in the interior of any facet or edge.

2.2 Definition of "being in contact with supports"

The *unit sphere*, i.e., the boundary of the three-dimensional ball centered at the origin and having radius one, is denoted by $\mathbb{S}^2$. We consider *directions* as points—or unit vectors—on $\mathbb{S}^2$. For any point $x \in \mathbb{R}^3$, and any direction $\mathbf{d} \in \mathbb{S}^2$, we denote by $r_{x\mathbf{d}}$ the ray emanating from x having direction $\mathbf{d}$.

We now formally define the notion of a point or facet *being in contact with supports* for a given build direction. It turns out to be convenient to distinguish three cases. Let F be a facet of $\mathcal{P}$, and $\mathbf{d} \in \mathbb{S}^2$ a direction. Let α be the angle between $\mathbf{d}$ and the outer normal of F. Note that $0 \leq \alpha \leq \pi$. If $\alpha < \pi/2$, then we say that F is a *front facet w.r.t.* $\mathbf{d}$. Similarly, if $\alpha > \pi/2$, then we say that F is a *back facet w.r.t.* $\mathbf{d}$. Finally, if $\alpha = \pi/2$, then we say that F is a *parallel facet w.r.t.* $\mathbf{d}$, or that $\mathbf{d}$ is *parallel to* F. Note that in the last case, the plane through the origin that is parallel to the plane through F contains the vector $\mathbf{d}$. We denote by P_F the great circle consisting of all directions that are parallel to facet F.

Definition 1 Let F be a facet of $\mathcal{P}$, x a point on F, and $\mathbf{d}$ a direction on $\mathbb{S}^2$. Point x is *in contact with supports for build direction* $\mathbf{d}$, if one of the following three conditions holds.

1. F is a back facet w.r.t. $\mathbf{d}$.

2. F is a front facet w.r.t. $\mathbf{d}$, and the ray $r_{x\mathbf{d}}$ intersects the boundary of $\mathcal{P}$ in a point that is not on facet F.

3. F is a parallel facet w.r.t. $\mathbf{d}$, and there is a facet G, $G \neq F$, such that
   (a) the ray $r_{x\mathbf{d}}$ intersects G, and
   (b) at least one of the vertices of G is strictly on the same side of the plane through F as the outer normal of F.

Definition 2 Let F be a facet of $\mathcal{P}$, and $\mathbf{d}$ a direction on $\mathbb{S}^2$.

1. We say that F is *in contact with supports for build direction* $\mathbf{d}$, if there is a point in the interior of F that is in contact with supports for build direction $\mathbf{d}$.

2. If F is not in contact with supports for build direction $\mathbf{d}$, then we say that F is *protected* from supports for build direction $\mathbf{d}$.

Note that for any direction $\mathbf{d} \in \mathbb{S}^2$, a back facet is always (completely) in contact with supports, whereas a parallel or front facet may be in contact with supports (possibly partially).

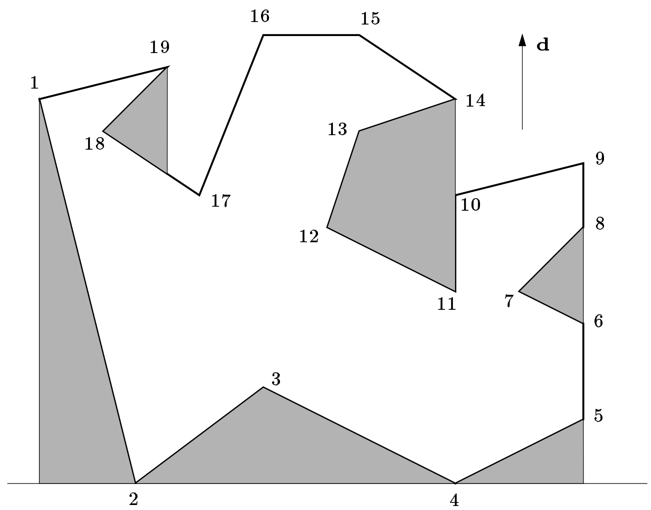

Figure 1: *Illustrating the two-dimensional variant of Definitions 1 and 2 for a planar simple polygon with* 19 *vertices. The shaded regions are the supports for the vertical build direction* **d**.

In Figure 1, we illustrate the two-dimensional variants of Definitions 1 and 2 for a planar simple polygon. Let **d** be the vertical build direction. Note that no interior point of the vertical edges $(5, 6)$ and $(8, 9)$ is in contact with supports. On the other hand, all points of the vertical edge $(10, 11)$ are in contact with supports. For build direction **d**, the edges $(5, 6)$, $(8, 9)$, $(9, 10)$, $(14, 15)$, $(15, 16)$, $(16, 17)$, and $(19, 1)$ are not in contact with supports, whereas the remaining edges are in contact with supports.

Definition 3 Let **d** be a direction on $\mathbb{S}^2$. The *contact area for build direction* **d** is defined as the total area occupied by all points on the boundary of $\mathcal{P}$ that are in contact with supports for build direction **d**.

The following lemma follows immediately from Definitions 1 and 2.

Lemma 1 *Let F be a facet of $\mathcal{P}$, and* **d** *a direction on $\mathbb{S}^2$. If F is in contact with supports for build direction* **d***, then there is a disk D in the interior of F having positive radius, such that every point $x \in D$ is in contact with supports for build direction* **d**.

Lemma 1 implies that if a facet F is in contact with supports for a given build direction, then it contributes a positive amount to the contact area.

2.3 Spherical convex hulls

In this paper, we will need two notions of convexity. A subset V of a Euclidean space is *convex*, if for all points p and q in V, the line segment pq is completely contained in V. The *convex hull* of a finite set of points is defined as the smallest convex set that contains these points. We will denote the convex hull of the points $p_1, p_2, \ldots, p_k$ by $CH(p_1, p_2, \ldots, p_k)$.

We generalize this notion to convexity on $\mathbb{S}^2$. (See also Chen and Woo [6].) Let $\mathbf{d}$ and $\mathbf{d}'$ be two distinct and non-antipodal points on $\mathbb{S}^2$, and let H be the (unique) plane in $\mathbb{R}^3$ that contains $\mathbf{d}$, $\mathbf{d}'$, and the origin. The intersection of this plane with $\mathbb{S}^2$ is a *great circle*. Along this great circle, there are two ways to walk from $\mathbf{d}$ to $\mathbf{d}'$. These two paths are called *great arcs*. The *geodesic* joining $\mathbf{d}$ and $\mathbf{d}'$ is the shorter of these two great arcs. If the points $\mathbf{d}$ and $\mathbf{d}'$ are antipodal, then all paths from $\mathbf{d}$ to $\mathbf{d}'$ along any great circle through these two points have the same length. In this case, we define the geodesic joining $\mathbf{d}$ and $\mathbf{d}'$ as the entire unit sphere. Finally, the geodesic joining a point $\mathbf{d}$ with itself is just point $\mathbf{d}$.

Let V be a subset of $\mathbb{S}^2$. We say that V is *spherically convex*, if for all points $\mathbf{d}$ and $\mathbf{d}'$ in V, the geodesic joining $\mathbf{d}$ and $\mathbf{d}'$ is completely contained in V.

The *spherical convex hull* of a finite set D of points on $\mathbb{S}^2$ is defined as the smallest spherically convex set that contains all points of D. Note that if D contains two antipodal points, the spherical convex hull of D is the entire unit sphere. We say that the set D is *hemispherical*, if there is a three-dimensional plane H through the origin, such that all elements of D are strictly on one side of H. If D is hemispherical, then the spherical convex hull of D is not the entire unit sphere. In this case, each edge of the spherical convex hull is a great arc.

We say that a point $\mathbf{d} \in \mathbb{S}^2$ is contained in the (spherical) convex hull C of a set of points, if it is in the interior or on the boundary of C.

If $p = (p_{\mathtt{x}}, p_{\mathtt{y}}, p_{\mathtt{z}})$ is a point in $\mathbb{R}^3$, then we denote its Euclidean length by $|p|$, i.e.,

$$|p| = \sqrt{p_{\mathtt{x}}^2 + p_{\mathtt{y}}^2 + p_{\mathtt{z}}^2}.$$

The following lemma shows a relation between our two notions of convexity.

Lemma 2 *Let S be a finite set of points in $\mathbb{R}^3$, and p a non-zero point in $\mathbb{R}^3$ that is contained in the convex hull of S. Let*

$$D := \{\mathbf{d} \in \mathbb{S}^2 : \exists q \in S, q \neq 0 \text{ and } \mathbf{d} = q/|q|\},$$

and $\mathbf{p}' := p/|p|$. Then $\mathbf{p}'$ is contained in the spherical convex hull of D.

Proof. Assume that $\mathbf{p}'$ is not contained in the spherical convex hull of D. We claim that there is a plane H through the origin that separates D from $\mathbf{p}'$. More precisely, $\mathbf{p}'$ is strictly on one side of H, whereas all elements of D are on the other side of H; there may be elements of D that are contained in H.

To prove this claim, first observe that it clearly holds if D consists of one element. So assume that $|D| \geq 2$. We may assume without loss of generality that $\mathbf{p}' = (0, 0, 1)$. If each element of D is on or below the plane $\mathbf{z} = 0$, then the claim holds. So assume that D contains one or more elements that are strictly above the plane $\mathbf{z} = 0$.

We denote the spherical convex hull of D by *SCH*. Let $\mathbf{x}$ be a point of *SCH* such that the distance on $\mathbb{S}^2$ between $\mathbf{p}'$ and $\mathbf{x}$ is minimum. Let *SD* be the spherical disk centered at $\mathbf{p}'$ and having $\mathbf{x}$ on its boundary. Note that *SD* has positive radius. Moreover, because of our choice of $\mathbf{x}$, and because $\mathbf{p}'$ is not in *SCH* by assumption, the interior of *SD* does not have any point in common with *SCH*.

Let H be the plane through the origin and $\mathbf{x}$ that is tangent to *SD*. Note that $\mathbf{p}' \notin H$. Then H is the plane we are looking for. Indeed, assume that there is an element $\mathbf{d} \in D$ such that $\mathbf{d}$ and $\mathbf{p}'$ are strictly on the same side of H. Let g be the geodesic joining $\mathbf{x}$ and $\mathbf{d}$. Then g is contained in *SCH*. Hence, there is a point $\mathbf{y}$ on g that is in the interior of *SD*. This is a contradiction.

Hence, we have proved the claim. Consider the plane H through the origin that separates D from $\mathbf{p}'$. All points of S are on one side of H, and p is strictly on the other side of H. Therefore, p is not contained in the convex hull of S. This is a contradiction. ∎

3 Protecting one facet from supports

Throughout this section, F denotes a fixed facet of the polyhedron $\mathcal{P}$. We consider the problem of computing a description of all build directions $\mathbf{d}$, such that facet F is not in contact with supports, when $\mathcal{P}$ is built in direction $\mathbf{d}$. Note that such a direction $\mathbf{d}$ may not exist.

3.1 The basic idea

We first describe our basic approach. For each facet G, $G \neq F$, we define a set $C_{FG} \subseteq \mathbb{S}^2$ of directions, such that for each $\mathbf{d} \in C_{FG}$, facet F is in contact with supports for build direction $\mathbf{d}$ "because of" facet G. That is, there is a point x in the interior of F, such that the ray $r_{x\mathbf{d}}$ emanating from

x and having direction $\mathbf{d}$ intersects facet G. Then, for each direction in the complement of the union of all sets C_{FG}, $G \neq F$, facet F is not in contact with supports. It will turn out, however, that we have to be careful with directions that are on the boundary of a set C_{FG}. Therefore, in the next subsection, we give a definition of the sets C_{FG} that does not use the notion of F being in contact with supports. In Sections 3.3 and 3.4, we give some properties of these sets, and show how they can be computed. Then, in Section 3.5, we show the relation between the sets C_{FG} and the question of when F is in contact with supports.

3.2 Definition of the sets C_{FG}

For any facet G of $\mathcal{P}$, and any point $x \in \mathbb{R}^3$, we define

$$R_{xG} := \{\mathbf{d} \in \mathbb{S}^2 : (r_{x\mathbf{d}} \cap G) \setminus \{x\} \neq \emptyset\}.$$

In words, if $\mathbf{d} \in R_{xG}$, then the ray from x in direction $\mathbf{d}$ intersects facet G in a point that is not equal to x. For any facet G of $\mathcal{P}$, we define

$$C_{FG} := \bigcup_{x \in F} R_{xG}.$$

Note that C_{FG} is closed, in the sense that its boundary belongs to the set itself.

Remark 1 At first sight, it seems more natural to define these sets as follows:

$$R'_{xG} := \{\mathbf{d} \in \mathbb{S}^2 : r_{x\mathbf{d}} \cap G \neq \emptyset\},$$

and

$$C'_{FG} := \bigcup_{x \in F} R'_{xG}.$$

If the facets F and G have a point, say x, in common, then for every $\mathbf{d} \in \mathbb{S}^2$, we have $r_{x\mathbf{d}} \cap G \neq \emptyset$, i.e., $C'_{FG} = \mathbb{S}^2$. Clearly, this is not what we want.

3.3 Some properties of the sets C_{FG}

In this section, we state some relatively easy properties of the sets C_{FG}.

Lemma 3 *If F and G are disjoint facets, or their intersection is a single point, then the set C_{FG} is hemispherical.*

Proof. First assume that F and G are disjoint. Let H be any plane such that F and G are strictly on different sides of H. (This plane exists because F and G are convex.) Let H' be the plane through the origin that is parallel to H. Then the set C_{FG} is strictly on one side of H', i.e., C_{FG} is hemispherical.

Next assume that F and G have exactly one point in common. Denote this common point by u. Then u must be a vertex of both facets. Let H be any plane that (i) contains u, but no other points of F, and no other points of G, and (ii) separates F and G. Let H' be the plane through the origin that is parallel to H. It follows from the definition of the sets R_{xG}, that no direction of C_{FG} is contained in H'. Then it follows easily that C_{FG} is strictly on one side of H'. Hence, also in this case, C_{FG} is hemispherical. ∎

Remark 2 In Lemma 3, we used the fact that F and G are triangular facets. If e.g. F is a facet with a hole, and G is inside this hole, then the set C_{FG} is not hemispherical.

Lemma 4 *If F and G are two distinct, but coplanar facets of $\mathcal{P}$, then C_{FG} is contained in a great circle on $\mathbb{S}^2$. In particular, the interior of C_{FG} is empty.*

Proof. Let H be the plane through the origin that is parallel to the plane through facet F. If x is any point on F, and $\mathbf{d}$ is any direction, then the ray $r_{x\mathbf{d}}$ intersects G only if it is contained in H. Therefore, C_{FG} is contained in the great circle that is obtained by intersecting H with $\mathbb{S}^2$. ∎

Lemma 5 *Let G be a facet that is not coplanar with F. Assume that F and G have an edge in common. Also, assume that for each vertex of G, one of the following is true: It is in the plane through F or on the same side of the plane through F as the outer normal of F. Consider the great circles P_F and P_G consisting of all directions that are parallel to F and G, respectively. Then C_{FG} is the set of all directions that are*

1. *on or on the same side of P_F as the outer normal of facet F, and*

2. *on or on the same side of P_G as the inner normal of facet G.*

Proof. The proof follows from the definition of the set C_{FG}. ∎

3.4 How to compute the sets C_{FG}

Let G be a facet of $\mathcal{P}$. Assume that either F and G are disjoint, or intersect in a single point. In the latter case, this point is a vertex of both facets. (We allow F and G to be coplanar.) In this section, we will show how the set C_{FG} can be computed.

First, we introduce some notation. Let $s = (s_{\mathtt{x}}, s_{\mathtt{y}}, s_{\mathtt{z}})$ and $t = (t_{\mathtt{x}}, t_{\mathtt{y}}, t_{\mathtt{z}})$ be two distinct points in $\mathbb{R}^3$, and let ℓ be the Euclidean length of the point

$$t - s := (t_{\mathtt{x}} - s_{\mathtt{x}}, t_{\mathtt{y}} - s_{\mathtt{y}}, t_{\mathtt{z}} - s_{\mathtt{z}}).$$

That is, ℓ is the Euclidean distance between $t - s$ and the origin. We will denote by $\mathbf{d}_{st}$ the point on $\mathbb{S}^2$ having the same direction as the directed line segment from s to t. That is, $\mathbf{d}_{st}$ is the point on $\mathbb{S}^2$ having coordinates

$$\mathbf{d}_{st} = \left(\frac{t_{\mathtt{x}} - s_{\mathtt{x}}}{\ell}, \frac{t_{\mathtt{y}} - s_{\mathtt{y}}}{\ell}, \frac{t_{\mathtt{z}} - s_{\mathtt{z}}}{\ell} \right).$$

The next lemma yields an algorithm for computing the set C_{FG}.

Lemma 6 *Let G be a facet of $\mathcal{P}$. Assume that F and G are disjoint or intersect in a single point. Let*

$$D_{FG} := \{\mathbf{d}_{st} \in \mathbb{S}^2 : s \textit{ is a vertex of } F,\ t \textit{ is a vertex of } G,\ s \neq t\}.$$

Then C_{FG} is the spherical convex hull of the at most nine directions in D_{FG}.

Proof. First note that $D_{FG} \subseteq C_{FG}$. We denote the spherical convex hull of the elements of D_{FG} by SCH_{FG}.

We will prove that (i) $C_{FG} \subseteq SCH_{FG}$, and (ii) C_{FG} is spherically convex. Since SCH_{FG} is the smallest spherically convex set that contains the elements of D_{FG}, the fact that $D_{FG} \subseteq C_{FG}$, together with (i) and (ii), imply that $C_{FG} = SCH_{FG}$.

To prove (i), let $\mathbf{d} \in C_{FG}$. We will show that $\mathbf{d} \in SCH_{FG}$. Since $\mathbf{d} \in C_{FG}$, there is a point x on F such that $\mathbf{d} \in R_{xG}$. Let y be any point such that $y \neq x$ and $y \in r_{x\mathbf{d}} \cap G$.

Let a, b, and c denote the three vertices of F, and u, v, and w the three vertices of G. Recall that $CH(a, b, c)$ denotes the (standard) convex hull of the points a, b, and c. Since $x \in CH(a, b, c)$, we have $y - x \in CH(y - a, y - b, y - c)$. Similarly, since $y \in CH(u, v, w)$, we have

$$y - a \in CH(u - a, v - a, w - a) =: CH_1,$$

$$y - b \in CH(u - b, v - b, w - b) =: CH_2,$$

and

$$y - c \in CH(u - c, v - c, w - c) =: CH_3.$$

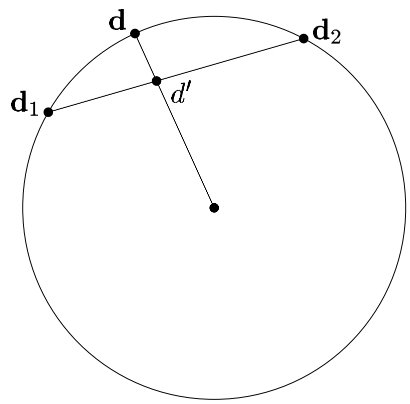

Figure 2: *Illustrating the proof of Lemma 6.*

Therefore,

$$CH(y-a, y-b, y-c) \subseteq CH(CH_1, CH_2, CH_3).$$

Hence, if we denote the (standard) convex hull of the point set[1]

$$\{t - s : s \text{ is a vertex of } F, t \text{ is a vertex of } G\}$$

by CH_{FG}, then we have shown that

$$y - x \in CH(CH_1, CH_2, CH_3) = CH_{FG}.$$

Since $y - x \neq 0$, and the ray from the origin through point $y - x$ has direction $\mathbf{d}$, Lemma 2 implies that $\mathbf{d} \in SCH_{FG}$. This completes the proof of (i).

It remains to prove that C_{FG} is spherically convex. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two distinct directions in C_{FG}, and let $\mathbf{d}$ be any direction on the shortest great arc that connects $\mathbf{d}_1$ and $\mathbf{d}_2$. (By Lemma 3, this arc is uniquely defined.) We have to show that $\mathbf{d} \in C_{FG}$.

Let H be the plane through $\mathbf{d}_1$, $\mathbf{d}_2$, and the origin. Then $\mathbf{d}$ is also contained in H. Let $d' \in \mathbb{R}^3$ be the intersection between the line through $\mathbf{d}_1$ and $\mathbf{d}_2$, and the line through the origin and $\mathbf{d}$. (Refer to Figure 2. By Lemma 3, d' is not the origin.) Let $0 \leq \lambda \leq 1$ be such that

$$d' = \lambda \mathbf{d}_1 + (1 - \lambda)\mathbf{d}_2.$$

Since $\mathbf{d}_1 \in C_{FG}$, there is a point x_1 on F such that the ray from x_1 in direction $\mathbf{d}_1$ intersects G in a point that is not equal to x_1. Let y_1 be any

[1] if F and G share a vertex, then this point set contains the origin.

such intersection point. Similarly, there is a point x_2 on F, and a point y_2 on G, such that $y_2 \neq x_2$ and the ray from x_2 in direction $\mathbf{d}_2$ intersects G in y_2.

Let α, α_1, and α_2 be the positive real numbers such that $\mathbf{d} = \alpha d'$, $y_1 - x_1 = \alpha_1 \mathbf{d}_1$, and $y_2 - x_2 = \alpha_2 \mathbf{d}_2$.[2] Then

$$\begin{aligned} \mathbf{d} &= \alpha(\lambda \mathbf{d}_1 + (1-\lambda)\mathbf{d}_2) \\ &= \frac{\alpha\lambda}{\alpha_1}(y_1 - x_1) + \frac{\alpha(1-\lambda)}{\alpha_2}(y_2 - x_2) \\ &= \left(\frac{\alpha\lambda}{\alpha_1} y_1 + \frac{\alpha(1-\lambda)}{\alpha_2} y_2\right) - \left(\frac{\alpha\lambda}{\alpha_1} x_1 + \frac{\alpha(1-\lambda)}{\alpha_2} x_2\right). \end{aligned}$$

Let μ be the real number such that

$$\mu \left(\frac{\alpha\lambda}{\alpha_1} + \frac{\alpha(1-\lambda)}{\alpha_2}\right) = 1.$$

Define

$$x := \frac{\mu\alpha\lambda}{\alpha_1} x_1 + \frac{\mu\alpha(1-\lambda)}{\alpha_2} x_2,$$

and

$$y := \frac{\mu\alpha\lambda}{\alpha_1} y_1 + \frac{\mu\alpha(1-\lambda)}{\alpha_2} y_2.$$

Then $\mu > 0$, and $\mu \mathbf{d} = y - x$. Our choice of μ implies that x is a convex combination of x_1 and x_2, and y is a convex combination of y_1 and y_2. It follows that x and y are points on the facets F and G, respectively. Moreover, $y \neq x$. Hence, the ray from x in direction $\mathbf{d}$ intersects G in a point that is not equal to x, i.e., $\mathbf{d} \in R_{xG}$. This shows that $\mathbf{d} \in C_{FG}$. ∎

3.5 More properties of the sets C_{FG}

In this section, we show the relation between the sets C_{FG} and the question whether facet F is in contact with supports.

We say that a facet G is *below* facet F, if each vertex of G is in the plane through F or on the same side of this plane as the inner normal of F. Hence, a facet G is *not below* F, if and only if at least one vertex of G is strictly on the same side of the plane through F as the outer normal of F.

Lemma 7 *Let* $\mathbf{d}$ *be a direction on* $\mathbb{S}^2$ *such that* F *is a front facet w.r.t.* $\mathbf{d}$. *Assume that* F *is in contact with supports for build direction* $\mathbf{d}$. *Then there is a facet* G*, such that*

[2] α_1 and α_2 exist, because $y_1 \neq x_1$ and $y_2 \neq x_2$.

1. *G is not below F, and*

2. $\mathbf{d}$ *is in the interior of C_{FG}.*

Proof. Assume without loss of generality that $\mathbf{d}$ is the vertically upwards direction. Hence, F is not vertical.

By Lemma 1, there is a disk D_0 in the interior of F, having positive radius, such that each point of D_0 is in contact with supports for build direction $\mathbf{d}$. That is, the vertical ray that emanates from any point of D_0 intersects the boundary of $\mathcal{P}$ in a point that is not on F. Thus, there is a disk D of positive radius in the interior of D_0, and a facet G, $G \neq F$, such that for each $x \in D$, the ray $r_{x\mathbf{d}}$ intersects G. Put differently, let $VC := \{r_{x\mathbf{d}} : x \in D\}$, i.e., VC is the vertical "cylinder" which is unbounded in direction $\mathbf{d}$ and is bounded from below by D. Since F is not vertical, the cylinder VC is not contained in a plane. If we move the disk D vertically upwards, then it stays in the cylinder VC, and each point of the moving disk passes through G. Clearly, facet G is not below F.

Let c and ϵ be the center and radius of D, respectively. Since D is not contained in a vertical plane, and $\epsilon > 0$, there is a spherical disk SD on $\mathbb{S}^2$, centered at $\mathbf{d}$ and having positive radius, such that for each $\mathbf{d}' \in SD$, the ray $r_{c\mathbf{d}'}$ intersects G in a point that is not equal to c. This implies that SD is completely contained in C_{FG}. Since $\mathbf{d}$ is in the interior of SD, it follows that $\mathbf{d}$ is in the interior of C_{FG}. ∎

We denote by $\mathcal{U}_F$ the union of the sets C_{FG}, where G ranges over all facets that are not below F.

The following two lemmas state when a front facet F is not in contact with supports.

Lemma 8 *Let $\mathbf{d}$ be a direction on $\mathbb{S}^2$ such that F is a front facet w.r.t. $\mathbf{d}$. If $\mathbf{d}$ is not in the interior of $\mathcal{U}_F$, then F is not in contact with supports for build direction $\mathbf{d}$.*

Proof. The claim follows immediately from Lemma 7. ∎

Lemma 9 *Let $\mathbf{d}$ be a direction on $\mathbb{S}^2$ such that F is a front facet w.r.t. $\mathbf{d}$. Assume that*

1. $\mathbf{d}$ *is in the interior of $\mathcal{U}_F$, and*

2. $\mathbf{d}$ *is not in the interior of C_{FG}, for every facet G that is not below F.*

Then F is not in contact with supports for build direction $\mathbf{d}$.

Proof. Again, this claim follows immediately from Lemma 7. ∎

Note that for any direction $\mathbf{d}$ that satisfies the conditions of Lemma 9, there are at least two facets G that are not below F and for which $\mathbf{d}$ is on the boundary of C_{FG}.

Lemma 10 *Let G be a facet of $\mathcal{P}$ that is not below F. Let $\mathbf{d}$ be a direction that is not parallel to F and that is in the interior of C_{FG}. Then F is in contact with supports for build direction $\mathbf{d}$.*

Proof. If F is a back facet w.r.t. $\mathbf{d}$, then, by definition, F is in contact with supports for build direction $\mathbf{d}$. So we may assume that F is a front facet w.r.t. $\mathbf{d}$.

We may assume without loss of generality that $\mathbf{d}$ is the vertically upwards direction. Hence, F is not vertical. Let $T := \{r_{x\mathbf{d}} : x \in F\}$, i.e., T is the vertical triangular prism which is unbounded in direction $\mathbf{d}$ and is bounded from below by F.

We claim that facet G and the interior of T have a non-empty intersection. Assume this claim is true. Then, let y be any point of G that is in the interior of T. Hence, y is not on F and not on any of the three unbounded trapezoids that form the vertical boundary of T. Let x be the projection of y on facet F in direction $-\mathbf{d}$. Then, x is in the interior of F, and the ray $r_{x\mathbf{d}}$ intersects the boundary of $\mathcal{P}$ in a point that is not on F. Hence, facet F is in contact with supports for build direction $\mathbf{d}$.

It remains to prove the claim. Assume that the intersection of G and the interior of T is empty. Since $\mathbf{d} \in C_{FG}$, there is a point x on F, such that $(r_{x\mathbf{d}} \cap G) \setminus \{x\} \neq \emptyset$. This point x must be on the boundary of F. Let y be any point of $(r_{x\mathbf{d}} \cap G) \setminus \{x\}$. Since F is not vertical, point y is not on F. There is a vertical plane H such that (i) $x \in H$, (ii) F is completely on and to the "left" of H, (iii) $y \in H$, and (iv) G is completely on and to the "right" of H. Facet G may be completely contained in H, but at least one vertex of F is to the left of H. But this implies that $\mathbf{d}$ is on the boundary of C_{FG}, a contradiction. ∎

In Lemmas 8, 9, and 10, we considered all directions that are not parallel to facet F. The next lemmas treat the remaining cases.

Lemma 11 *Let $\mathbf{d}$ be a direction on the great circle P_F that is not contained in any of the sets C_{FG}, where G ranges over all facets that are not below F. Then facet F is not in contact with supports for build direction $\mathbf{d}$.*

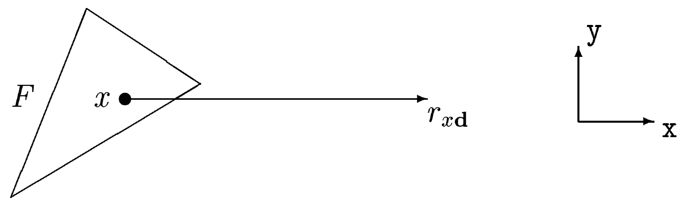

Figure 3: *Illustrating the proof of Lemma 11.*

Proof. We may assume without loss of generality that (i) P_F is contained in the plane $\mathsf{z} = 0$, (ii) $\mathbf{d}$ is the direction $(1, 0, 0)$, and (iii) the outer normal of F is the vector $(0, 0, 1)$.

Assume that F is in contact with supports for build direction $\mathbf{d}$. Then by Definitions 1 and 2, there is a point x in the interior of F, and a facet G, such that at least one vertex of G is strictly above the plane $\mathsf{z} = 0$, and the ray $r_{x\mathbf{d}}$ intersects G. (Refer to Figure 3.)

Since $\mathbf{d}$ is not contained in the set C_{FG}, there is a spherical disk SD on $\mathbb{S}^2$, centered at $\mathbf{d}$ and having positive radius, such that no direction of SD is contained in C_{FG}.

The facts that (i) at least one vertex of G is strictly above the plane $\mathsf{z} = 0$, and (ii) the ray $r_{x\mathbf{d}}$ intersects G, imply that there is an $\epsilon > 0$ and a real number δ, such that the unit-vector $\mathbf{d}'$ corresponding to the direction $(1, \delta, \epsilon)$ is contained in SD, and the ray $r_{x\mathbf{d}'}$ intersects G. (In words, we can rotate the direction $\mathbf{d}$ by a small amount such that it stays in the spherical disk SD, is strictly above the plane $\mathsf{z} = 0$, and the ray from x in the resulting direction still intersects facet G.)

Since x is in the interior of F, the ray $r_{x\mathbf{d}'}$ intersects G in a point that is not equal to x. Hence, the direction $\mathbf{d}'$ is contained in C_{FG}. This is a contradiction. ∎

Lemma 12 *Let G be a facet that is not below F. Let $\mathbf{d}$ be a direction on the great circle P_F, such that*

1. *either $\mathbf{d}$ is in the interior of the set C_{FG},*
2. *or $\mathbf{d}$ is in the interior of an edge of C_{FG}, and this edge is contained in P_F.*

Then facet F is in contact with supports for build direction $\mathbf{d}$.

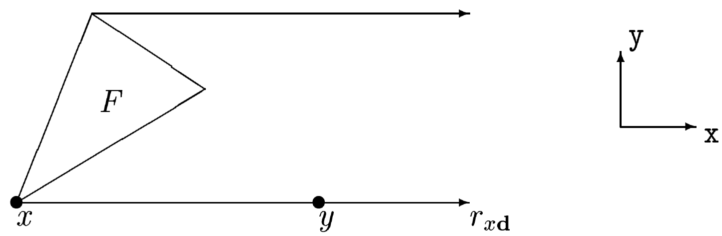

Figure 4: *Illustrating the proof of Lemma 12.*

Proof. We may assume without loss of generality that (i) P_F is contained in the plane $\mathsf{z} = 0$, (ii) $\mathbf{d}$ is the direction $(1, 0, 0)$, and (iii) the outer normal of F is the vector $(0, 0, 1)$.

The conditions of the lemma imply that there is a spherical disk SD on $\mathbb{S}^2$, centered at $\mathbf{d}$ and having positive radius, such that each direction that is contained in SD and that has a positive z-coordinate is contained in C_{FG}.

Assume that F is not in contact with supports for build direction $\mathbf{d}$. Then, by Definitions 1 and 2,

$$\text{for all } a \text{ in the interior of } F\text{: } r_{a\mathbf{d}} \cap G = \emptyset. \tag{1}$$

Since $\mathbf{d} \in C_{FG}$, there is a point x on F, such that the ray $r_{x\mathbf{d}}$ intersects G in a point that is not equal to x. It follows from (1), that each such point x is on the boundary of F. Moreover, (1) implies that for each such point x, the line through $r_{x\mathbf{d}}$ does not intersect the interior of F. We choose such a point x which is a vertex of F, and such that the ray $r_{x\mathbf{d}}$ only has point x in common with facet F. (If the line through $r_{x\mathbf{d}}$ contains an edge of F, then we take for x the "last" point on F in direction $\mathbf{d}$.) Let y be any point of $(r_{x\mathbf{d}} \cap G) \setminus \{x\}$. Hence, $y \in G$, and $y \notin F$. (Refer to Figure 4.)

Consider the set $W := \{r_{a\mathbf{d}} : a \in F\}$. This set is bounded by one or two edges of F and two lines that are in the plane $\mathsf{z} = 0$ and parallel to $\mathbf{d}$. Moreover, either x and y are both on the upper bounding line, or both on the lower bounding line. Assume without loss of generality that x and y are on the lower bounding line.

It follows from (1) and the facts that $y \in G$ and $y \notin F$, that F and G are disjoint. Moreover, there are $\epsilon > 0$ and $\delta > 0$, such that (i) the unit-vector $\mathbf{d}'$ corresponding to the direction $(1, \epsilon, \delta)$ is contained in the spherical disk SD, and (ii) for all points $a \in F$, the ray $r_{a\mathbf{d}'}$ does not intersect G. In particular, for each point $a \in F$, we have $(r_{a\mathbf{d}'} \cap G) \setminus \{a\} = \emptyset$. (In words, we can rotate the direction $\mathbf{d}$ by a small amount such that it stays in the spherical disk

SD, is strictly above the planes $\mathtt{z} = 0$ and $\mathtt{y} = 0$, and for all points $a \in F$, the ray from a in the resulting direction does not intersect facet G.)

This proves that the direction $\mathbf{d}'$ is not contained in the set C_{FG}. This is a contradiction, because the part of SD having a strictly positive $\mathtt{z}$-coordinate is contained in C_{FG}. ∎

Let $\mathcal{A}$ be the arrangement on $\mathbb{S}^2$ defined by the great circle P_F and the boundaries of the sets C_{FG}, where G ranges over all facets that are not below F.

Lemmas 11 and 12 do not consider directions on P_F that are vertices of $\mathcal{A}$. For these directions, we will use the following lemma to decide if F is in contact with supports.

Let $\mathbf{d}$ be a direction that is parallel to facet F, and let $W := \{r_{x\mathbf{d}} : x \in F\}$. The boundary of this set W consists of one or two edges of F, and two unbounded rays. A point is in the interior of W if it is contained in W, but is not on the boundary of W.

Lemma 13 *Let* $\mathbf{d}$ *be a direction on the great circle* P_F*,* $W := \{r_{x\mathbf{d}} : x \in F\}$*, and* $\mathcal{I}$ *the set of all facets* G *such that*

1. *G is not below F, and*
2. *the intersection of G and the interior of W is non-empty.*

Then F is in contact with supports for build direction $\mathbf{d}$ *if and only if* $\mathcal{I} \neq \emptyset$.

Proof. The proof follows from Definitions 1 and 2. ∎

3.6 The facet protection algorithm

The algorithm that computes a description of all build directions for which facet F is not in contact with supports is based on the previous results. Recall that the basic approach consists of computing the complement of the union of the sets C_{FG}, where G ranges over all facets of $\mathcal{P}$ that are not below F. It follows from Lemmas 9, 11, 12, and 13 that we have to be careful with directions that are on the boundary of a set C_{FG} or on the great circle P_F.

Step 1: Following Lemma 5, we do the following. For each facet G of $\mathcal{P}$ that is not below F, and that has an edge with F in common, compute the boundary of the set C_{FG} as the set of all directions that are on or on the same side of P_F as the outer normal of facet F, and on or on the same side of P_G as the inner normal of facet G.

Step 2: Following Lemma 6, we do the following. For each facet G of $\mathcal{P}$ that is not below F, and such that

1. either F and G are disjoint, or

2. F and G intersect in a single point,

do the following. Compute the boundary of the set C_{FG} as the spherical convex hull of the (at most nine) directions $\mathbf{d}_{st} \neq 0$, where s and t are vertices of F and G, respectively. (For a spherical convex hull algorithm, see e.g. Chen and Woo [6].) Note that by Lemma 3, this spherical convex hull is not the entire unit sphere.

Step 3: Compute the arrangement $\mathcal{A}$ on $\mathbb{S}^2$ that is defined by the great circle P_F and the bounding edges of all sets C_{FG} that were computed in Steps 1 and 2. Let $\mathcal{B}$ be the arrangement on $\mathbb{S}^2$ consisting of all vertices, edges, and faces of $\mathcal{A}$ that are on P_F or on the same side of P_F as the outer normal of F. Give each edge e of $\mathcal{B}$ an orientation, implying the notions of being to the "left" and "right" of e. For edges that are not contained in the great circle P_F, these orientations are chosen arbitrarily. For each edge e of $\mathcal{B}$ that is contained in P_F, we choose the orientation such that the outer normal of F is to the "left" of e.

For each edge e of $\mathcal{B}$, compute the following three values:

1. l_e, which is one if the interior of the face of $\mathcal{B}$ to the left of e is contained in some set C_{FG} that was computed in Step 1 or 2, and zero otherwise.

2. r_e, which is one if the interior of the face of $\mathcal{B}$ to the right of e is contained in some set C_{FG} that was computed in Step 1 or 2, and zero otherwise. If e is contained in P_F, then the value of r_e is not defined.

3. i_e, which is one if the interior of e is in the interior of some set C_{FG} that was computed in Step 1 or 2, and zero otherwise.

Moreover, for each vertex v of $\mathcal{B}$ that is not on P_F, compute the value i_v, which is one if v is in the interior of some set C_{FG} that was computed in Step 1 or 2, and zero otherwise.

We will show in Section 3.6.1 how Step 3 can be implemented.

Step 4: Select all edges e of $\mathcal{B}$ that are not contained in P_F, and for which $l_e = 1$ and $r_e = 0$, or $l_e = 0$ and $r_e = 1$. Also, select all edges e of $\mathcal{B}$ that are contained in P_F, and for which $l_e = 0$ and $i_e = 0$.

By Lemmas 8 and 11, these edges define polygonal regions on $\mathbb{S}^2$ that represent build directions for which facet F is not in contact with supports.

Step 5: Select all edges e of $\mathcal{B}$ that are not contained in P_F, and for which $l_e = r_e = 1$ and $i_e = 0$. Similarly, select all vertices v of $\mathcal{B}$ that are not on

P_F, for which $i_v = 0$, and having the property that $l_e = r_e = 1$ for all edges e of $\mathcal{B}$ that have v as one of their vertices.

By Lemma 9, these vertices and the interiors of these edges represent build directions for which facet F is not in contact with supports.

Step 6: Let D be the set of all vertices of $\mathcal{B}$ that are on P_F. For each direction $\mathbf{d} \in D$, decide if facet F is in contact with supports for build direction $\mathbf{d}$. This can be done by using an algorithm that is immediately implied by Lemma 13.

This algorithm reports a collection of spherical polygons, great arcs (the edges computed in Step 5), and single directions (the vertices computed in Steps 5 and 6). It follows from the previous results that this collection represents all build directions for which facet F is not in contact with supports.

The implementation of the algorithm is straightforward, except for Step 3, which we now consider in more detail.

3.6.1 Implementing Step 3

After Steps 1 and 2, we have a collection of at most $n-1$ spherical polygons, each having $O(1)$ edges. Note that each edge of such a polygon is a great arc. For each such edge e, let K_e be the great circle that contains e. Using an incremental algorithm, we compute the arrangement $\mathcal{A}'$ on $\mathbb{S}^2$ of the $O(n)$ great circles K_e, and the great circle P_F. (In de Berg et al.[7], an incremental algorithm is given that computes the arrangement of lines in the plane. Via central projection [6], this algorithm can be "translated" such that it computes $\mathcal{A}'$.)

By removing from $\mathcal{A}'$ all vertices and edges that are strictly on the same side of P_F as the inner normal of facet F, we obtain an arrangement which we denote by $\mathcal{B}'$. We give each edge of $\mathcal{B}'$ a direction. We will show how the values l_e, r_e, i_e, and i_v for all edges e and vertices v of $\mathcal{B}'$ can be computed. Since the arrangement $\mathcal{B}$ is obtained from $\mathcal{B}'$ by removing all vertices and edges that are not contained in edges of our original polygons C_{FG}, this solves our problem.

We introduce the following notation. For each vertex v of $\mathcal{B}'$, let I_v be the set of all facets G that are not below F and for which v is in the interior of C_{FG}. For each edge e of $\mathcal{B}'$, let L_e be the set of all facets G that are not below F and for which the interior of the face of $\mathcal{B}'$ to the left of e is contained in C_{FG}. Similarly, let R_e be the set of all facets G that are not below F and for which the interior of the face of $\mathcal{B}'$ to the right of e is contained in C_{FG}. Clearly,

1. $i_v = 1$ if and only if $I_v \neq \emptyset$,

2. $l_e = 1$ if and only if $L_e \neq \emptyset$,

3. $r_e = 1$ if and only if e is not contained in P_F and $R_e \neq \emptyset$, and

4. $i_e = 1$ if and only if $L_e \cap R_e \neq \emptyset$.

The idea is to traverse each great circle that defines the arrangement $\mathcal{B}'$, and maintain the sizes of the sets I_v, L_e, R_e, and $L_e \cap R_e$. We number the facets of $\mathcal{P}$ arbitrarily from 1 to n.

Let K be any of the great circles that define $\mathcal{B}'$, and let v be a vertex of $\mathcal{B}'$ which is on K. By considering all facets G that are not below F, we compute the set I_v, and store it as a bit-vector I of length n. Note that since C_{FG} has at most nine vertices, deciding if v is in the interior of C_{FG} can be done by brute force, in $O(1)$ time. By traversing this array I, we compute the number of ones it contains, and deduce from this number the value i_v.

Let e be an edge of $\mathcal{B}'$ which is contained in K and has v as a vertex. By considering all edges of $\mathcal{B}'$ that have v as endpoint, we know which sets C_{FG} are entered or left, when our traversal along K leaves v and enters the interior of e. We make two copies of the bit-vector I, and call them L and R. Then, by flipping the appropriate bits in the three arrays L, R, and I, we obtain the bit-vectors for the sets L_e, R_e, and $L_e \cap R_e$, respectively, and the number of ones they contain. This gives us the values l_e, r_e, and i_e.

We now continue our traversal of the great circle K. Each time we reach or leave a vertex of $\mathcal{B}'$, we flip the appropriate bits in the arrays L, R, and I, and deduce the l, r, and i values.

Theorem 1 *Let $\mathcal{P}$ be a polyhedron with n triangular facets, and let F be a facet of $\mathcal{P}$. In $O(n^2)$ time and using $O(n^2)$ space, we can compute a description of all build directions for which F is not in contact with supports.*

Proof. The correctness of the algorithm given above follows from the results presented in Sections 3.4 and 3.5. So it remains to analyze the running time and the amount of space used.

Clearly, Step 1 takes $O(n)$ time. We know from Lemma 6, that for each facet G that is considered in Step 2, C_{FG} is the spherical convex hull of at most nine points. Therefore, the boundary of C_{FG} can be computed in $O(1)$ time. It follows that Step 2 takes $O(n)$ time.

The *zone theorem* [7] implies that the arrangement $\mathcal{A}'$ can be computed in $O(n^2)$ time. (Note that the zone theorem is stated for arrangements of lines in the plane. Via central projection, it is easy to see that it also holds for great circles.)

Given $\mathcal{A}'$, we obtain the arrangement $\mathcal{B}'$ in $O(n^2)$ time. Note that $\mathcal{B}'$ is defined by $O(n)$ great circles. Let K be any of these great circles, and let v be a vertex of $\mathcal{B}'$ which is on K. It takes $O(n)$ time to compute the set I_v, and the bit-vectors I, L, and R. It follows from the zone theorem that by traversing K, we perform $O(n)$ insertions and deletions in these arrays. Each insertion and deletion is performed by flipping one bit and, hence, can be done in $O(1)$ time. Hence, the total time for this great circle K is bounded by $O(n)$. We conclude that we can implement Step 3 in $O(n^2)$ time.

Steps 4 and 5 can easily be done in $O(n^2)$ time. Consider Step 6. By the zone theorem, the set D contains $O(n)$ directions. Then, Lemma 13 implies that we can implement Step 6 in $O(n^2)$ total time.

Since the entire algorithm takes $O(n^2)$ time, it uses $O(n^2)$ space. ∎

Remark 3 Our algorithm computes a description of all build directions for which facet F is not in contact with supports. If we want to compute only one such direction, then we can use the *topological walk* algorithm of Asano et al. [2]. This reduces the space bound to $O(n)$. The time bound, however, remains $O(n^2)$. Note that the topological walk does not reduce the storage requirements if we wish to compute (as we have) a description of the set of all directions for which a facet F is not in contact with supports, because such a description can have $\Omega(n^2)$ connected components. (See Remark 4 below.)

Remark 4 We claim that the set of all build directions for which a facet F is not in contact with supports can have $\Omega(n^2)$ connected components. Consider the polygonal line in Figure 5. Let it be contained in the plane $\mathsf{z} = 0$. Starting at the endpoint in the center, follow the line, and move each vertex a little bit higher (in the z-direction) than the previous one. Then, by "thickening" the line segments by a small amount, we get a snake-like polyhedron. If n denotes the number of facets, then a linear number of them span the left-to-right range, and a linear number span the bottom-to-top range. Let F be the small triangular facet that is obtained by thickening the endpoint in the center. For this facet, our claim holds.

Remark 5 The algorithm can be extended to the case when k facets, $F_1, F_2, \ldots, F_k$ have to be protected from being in contact with supports. For each F_i, we run Steps 1 and 2 of the algorithm in Section 3.6. This gives us a set of spherically convex regions of total size $O(kn)$. We then compute the arrangement of these regions and the k great circles P_{F_i}, which has size $O(k^2n^2)$, and then traverse it in essentially the way described in the algorithm. The running time is bounded by $O(k^2n^2)$.

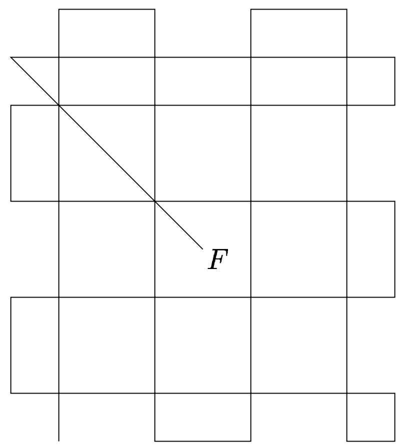

Figure 5: *The set of all build directions for which facet F is not in contact with supports can have a quadratic number of connected components.*

3.7 Experimental results

We have implemented a simplified version of the algorithm of Theorem 1. In this implementation, the boundary of the union $\mathcal{U}_F$ is computed incrementally, i.e., the sets C_{FG}, where G ranges over all facets that are not below F, are added one after another in a brute force manner. For simplicity, we ignored directions in the interior of $\mathcal{U}_F$ for which the condition of Lemma 9 holds. Also, directions on the great circle P_F were ignored in our implementation. The program outputs a collection of spherical polygons on $\mathbb{S}^2$ such that for each direction in such a polygon, facet F is not in contact with supports. Hence, the program finds, in general, a subset of all directions for which facet F is not in contact with supports. More details of the implementation can be found in [17].

The program is written in C++ using LEDA 3.8 [14]. We have tested our implementation on real-world polyhedral models obtained from Stratasys, Inc., on a SUN Ultra (300 MHz, 512 MByte RAM). Although the running time of our implementation is $\Theta(n^3)$ in the worst case, the actual running time is reasonable in practice.

Table 1 gives some test results for five polyhedral models. (Additional test results can be found in [17].) For each model, we ran the program for different facets F, and averaged over the number of runs. These models are: (i) `rd_yelo.stl`, a long rod, with grooves cut along its length. The two ends of the rod are decagons. (ii) `cover-5.stl`, see Figure 6. (iii) `tod21.stl`, see Figure 7. (iv) `stlbin2.stl`, an open rectangular box, with a hole on each

model	n	$\#F$	$\#C_{FG}$	$\lvert\mathcal{U}_F\rvert$	min	max	average
rd_yelo.stl	396	396	99	3.5	0.01	73	16
cover-5.stl	906	906	482	8.2	0.03	558	103
tod21.stl	1,128	1,128	229	3.8	0.05	281	25
stlbin2.stl	2,762	1,330	1178	20.9	0.25	2,019	363
mj.stl	2,832	1,000	641	10.1	0.26	2,270	146

Table 1: *Some performance numbers for our implementation. n denotes the number of facets of the model; $\#F$ denotes the number of facets F for which we ran the program independently and averaged our bounds over; $\#C_{FG}$ denotes the average number of facets G that are not below F; $\lvert\mathcal{U}_F\rvert$ denotes the average number of vertices on the boundary of the union $\mathcal{U}_F$ (note that this union may have no vertices at all);* min, max, *and* average *denote the minimum, maximum, and average time in seconds. The minimum occurs when all facets G are below F. The standard deviation is very large because the running time depends heavily on the choice of the facet F.*

side and interior flanges at the corners. (v) `mj.stl`, see Figure 8. (These models are displayed using the QuickSlice software front-end of Stratasys, Inc.)

4 Maximizing the total area of facets having no contact with supports

In this section, we consider the problem of computing a description of all build directions for which the sum of the areas of those facets that are not in contact with supports is maximum. We will see that this problem can be solved using the results from Section 3.

For each facet F of $\mathcal{P}$, we compute the sets C_{FG} for all facets G that are not below F. Recall that the boundary of C_{FG} consists of at most nine edges, which are great arcs.

Let $\mathcal{C}$ be the arrangement on $\mathbb{S}^2$ defined by

1. the great circles P_F, and

2. the great circles that contain an edge of some set C_{FG}.

Note that $\mathcal{C}$ is defined by $O(n^2)$ great circles and, hence, consists of $O(n^4)$ vertices, edges, and faces.

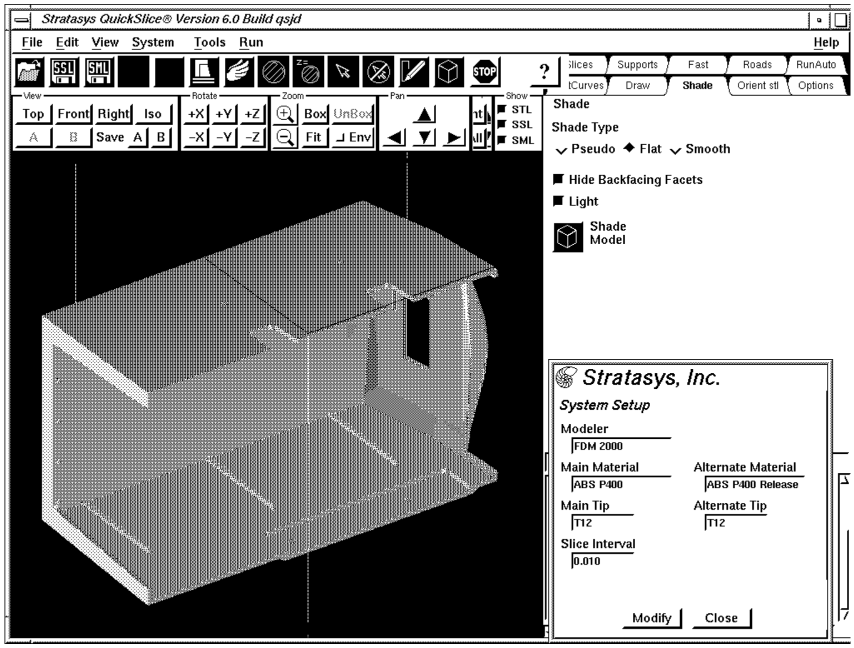

Figure 6: `cover-5.stl`

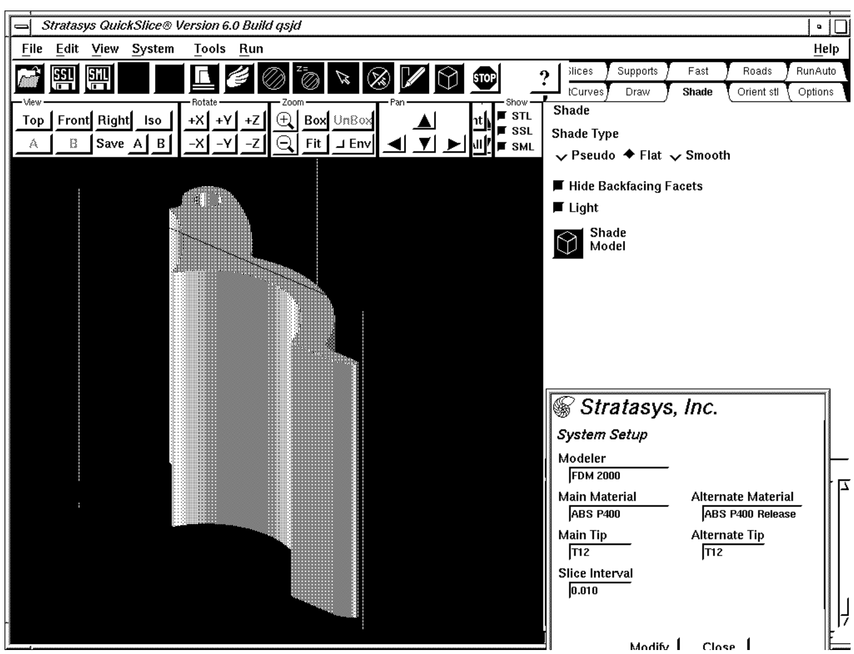

Figure 7: `tod21.stl`

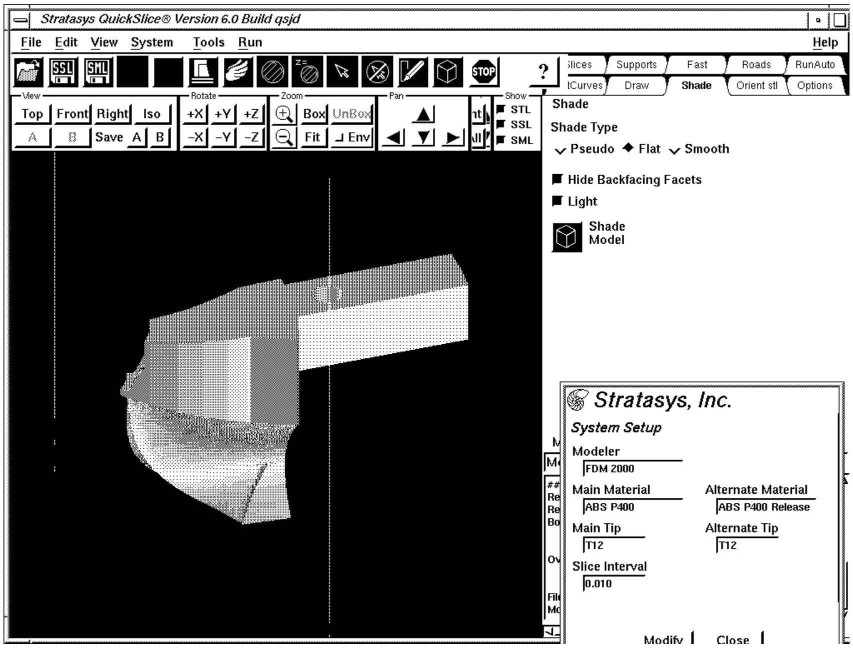

Figure 8: `mj.stl`

Lemma 14 *Let f be any face or edge of the arrangement $\mathcal{C}$. For all directions in the interior of f, the total area of those facets that are not in contact with supports is the same.*

Proof. Let $\mathbf{d}$ and $\mathbf{d}'$ be two directions in the interior of f. The results of Section 3 and the definition of the arrangement $\mathcal{C}$ imply that a facet F of $\mathcal{P}$ is in contact with supports for build direction $\mathbf{d}$, if and only if it is in contact with supports for build direction $\mathbf{d}'$. ■

By traversing each of the $O(n^2)$ great circles that define $\mathcal{C}$, we compute for each vertex (resp. edge and face) of $\mathcal{C}$, the total area of all facets of $\mathcal{P}$ that are not in contact with supports, when $\mathcal{P}$ is built in a direction that is equal to this vertex (resp. that is contained in this edge or face). This can be done by a straightforward generalization of the algorithm given in Section 3.6.1. Then by traversing $\mathcal{C}$, we select all vertices, edges, and faces for which the corresponding area is maximum.

Theorem 2 *Let $\mathcal{P}$ be a polyhedron with n triangular facets. In $O(n^4)$ time and using $O(n^4)$ space, we can compute a description of all build directions for which the sum of the areas of those facets having no contact with supports is maximum.*

Proof. The proof follows from the discussion above. ∎

5 Further work and concluding remarks

Consider the problem of computing a build direction for which the contact area is minimum. By generalizing the techniques in [11], we can reduce this problem to $O(n^6)$ optimization problems of the following type:

- minimize a function which is the sum of $O(n^2)$ fractions p_i/q_i, where
  - p_i is a polynomial of degree four in the variables d_1, d_2 and d_3, where $d_1^2 + d_2^2 + d_3^2 = 1$,
  - q_i is a polynomial of degree six in the variables d_1, d_2 and d_3, where $d_1^2 + d_2^2 + d_3^2 = 1$,
- subject to $O(n^3)$ linear constraints for d_1, d_2 and d_3.

Clearly, this is not a practical approach. Also, it is not clear if such an optimization problem can be solved in polynomial time. We are not aware of any efficient algorithm that minimizes the contact area for non-convex polyhedra. Therefore, it is natural to look for heuristics that approximate the minimum contact area for general polyhedra.

In Section 4, we showed how to compute all build directions for which the total area of all facets that are not in contact with supports is maximum. Clearly, this is equivalent to computing all build directions, for which the total area of all facets that are in contact with supports is minimum. In general, such a direction does not minimize the contact area, because some facets may be only partially in contact with supports. Our algorithm can, however, be used in a heuristic for approximating the smallest possible contact area.

For a given build direction, there are three types of facets: (i) facets that are not in contact with supports, (ii) facets that are completely in contact with supports, and (iii) facets that are partially in contact with supports. The contact area is determined by the facets of types (ii) and (iii). Until now, we did not distinguish between these two types.

The idea of our heuristic is as follows. Imagine that we divide each facet of $\mathcal{P}$ into triangles having an infinitesimally small perimeter. This gives a new polyhedron, which we denote by $\mathcal{P}'$. Consider any facet F of $\mathcal{P}'$. Since F is basically a point, it follows that if F is in contact with supports for some build direction, then every point on F is in contact with supports. Also, for this direction, $\mathcal{P}$ and $\mathcal{P}'$ have the same contact area. Hence, if $\mathbf{d}$ is a build direction that minimizes the total area of all facets of $\mathcal{P}'$ that are in contact with supports (or, equivalently, maximizes the total area of all facets of $\mathcal{P}'$ that are not in contact with supports), then this same direction minimizes the contact area of our original polyhedron P.

The heuristic consists of dividing each facet of $\mathcal{P}$ into triangles. Some possible strategies are (i) to divide each facet into the same number of triangles, or (ii) to divide "large" facets into a "large" number of triangles, and "small" facets into a "small" number of triangles.

We conclude this paper with some open problems. We have shown that for a fixed facet F of the polyhedron $\mathcal{P}$, a description of all build directions for which F is not in contact with supports can be computed in $O(n^2)$ time. We have also seen that this problem has an $\Omega(n^2)$ lower bound, because the output can have quadratic size. A natural question is to ask for the time complexity of computing *one* build direction for which F is not in contact with supports, or decide that such a direction does not exist. This problem appears to be closely related to the following one: Given $n+1$ triangles $T_0, T_1, T_2, \ldots, T_n$ in the plane, decide if T_0 is contained in the union of $T_1, \ldots, T_n$. It is known that this problem is 3SUM-hard, see Gajentaan and Overmars [8]. Therefore, we conjecture that computing a single direction for which facet F is not in contact with supports is 3SUM-hard as well.

We have shown that the algorithm of Section 3.6 is worst-case optimal. What about the algorithm of Section 4? That is, is there a polyhedron, such that the set of all directions for which the total area of all facets that are not in contact with supports is minimum, has $\Omega(n^4)$ connected components?

The algorithms of Sections 3.6 and 4 have running time $O(n^2)$ and $O(n^4)$, respectively. It would be interesting to design output-sensitive algorithms for solving these problems.

Finally, is it possible to compute, in polynomial time, a build direction for which the contact area is minimum?

Acknowledgements

We thank Larry Roscoe, Don Holzwarth, Jon Holt, Jeff Kotula, and Tom Studanski of Stratasys, Inc. for useful discussions. In particular, they proposed Problem 1 to us. We also thank them for allowing us to test our implementation on their polyhedral models and to use their software front-end, Quickslice, to visualize these models. We thank the anonymous referee for detailed comments, which helped improving the presentation.

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